ON THE STRUCTURE OF SOME REDUCED AMALGAMATED FREE PRODUCT C^*-ALGEBRAS

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ABSTRACT. We study some reduced free products of C^*-algebras with amalgamations. We give sufficient conditions for the positive cone of the K_0 group to be the largest possible. We also give sufficient conditions for simplicity and uniqueness of trace. We use the later result to give a necessary and sufficient condition for simplicity and uniqueness of trace of the reduced C^*-algebras of the Baumslag-Solitar groups $BS(m, n)$.

1. Introduction

In [17] Voiculescu indroduced the noncommutative probabilistic theory of freeness together with the notion of reduced amalgamated free products of C^*-algebras. The simplest case is amalgamation over the complex numbers, which was considered independently by Avitzour in [2]. Avitzour also gave a sufficient condition for simplicity and uniqueness of trace in the case of amalgamation over the complex numbers, which we generalize here using extensively his ideas. Avitzour's work is based on the work of Powers [14], in which Powers proved that the reduced C^*-algebra of the free group on two generators is simple and has a unique trace. Subsequently Pashke and Salinas in [12] and Choi in [3] considered other reduced C^*-algebras of amalgams of discrete groups. The most general result for the case of reduced C^*-algebras of amalgams of discrete groups, that generalize Power's result is due to de la Harpe ([9]). It is a corollary of our result. Our result is applicable to some HNN extensions of groups.

In [1] Anderson, Blackadar and Haagerup studied the scale and the positive cone of $\mathbf{K_0}$ for the Choi algebras. In [6] Dykema and Rørdam extended their result to the case of reduced free products of C^*-algebras (with amalgamation over the complex numbers). Using similar techniques we generalize the later result for the positive cone of $\mathbf{K_0}$ to the case of reduced amalgamated free products. The K-theory of reduced free products of nuclear C^*-algebras was determined by Germain in [7] in terms of the K-theory of the underlying C^*-algebras. He gave partial results in [8] for the K-theory of some reduced amalgamated free products. The question of determinig the K-theory of reduced C^*-algebras of amalgams of discrete groups in terms of the K-theory of the reduced C^*-algebras of the underlying groups was resolved completely by Pimsner in [13].

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2. The Construction of the Reduced Amalgamated Free Product and Preliminaries

In this section we will explain the construction of reduced amalgamated free products of C^*-algebras of Voiculescu, following closely [5, §1].

First we recall the definition of freeness. Suppose that we have unital C^*-algebras $1_{\mathfrak{A}} \in \mathfrak{B} \subset \mathfrak{A}$ and conditional expectation $\mathfrak{E} : \mathfrak{A} \to \mathfrak{B}$. Suppose that we have a family $\mathfrak{B} \subset \mathfrak{A}_\iota \subset \mathfrak{A}$, $\iota \in I$ of C^*-subalgebras of $\mathfrak{A}$, all of them containing $\mathfrak{B}$. We say that the family $\{\mathfrak{A}_\iota | \iota \in I\}$ is $\mathfrak{E}$-free if for any elements $a_k \in \mathfrak{A}_{\iota_k}$, $k = 1, \ldots, n$, such that $\iota_1 \neq \iota_2, \iota_2 \neq \iota_3, \ldots, \iota_{n-1} \neq \iota_n$ and $\mathfrak{E}(a_k) = 0$, we have $\mathfrak{E}(a_1 a_2 \cdots a_n) = 0$. We say that the elements $a_i \in \mathfrak{A}, i = 1, \ldots, n$ are $\mathfrak{E}$-free if the family $\{C^*(\mathfrak{B} \cup \{a_i\}) |\ i = 1, \ldots, n\}$ is $\mathfrak{E}$-free. This includes the case $\mathfrak{B} = \mathbb{C}$ and $\mathfrak{E}$ being a state.

Let I be a index set, card$(I) \geq 2$. Let B be a unital C^*-algebra and for each $\iota \in I$ we have a unital C^*-algebra A_ι, which contains a copy of B as a unital C^*-subalgebra. We also suppose that for each $\iota \in I$ there is a conditional expectation $E_\iota : A_\iota \to B$, satisfying

$$\forall a \in A_\iota,\ a \neq 0,\ \exists x \in A_\iota,\ E_\iota(x^* a^* a x) \neq 0. \tag{1}$$

The reduced amalgamated free product of (A_ι, E_ι) is denoted by

$$(A, E) = \mathop{*}_{\iota \in I} (A_\iota, E_\iota).$$

We will be mainly interested in the case of $B \neq \mathbb{C}$ and in this case the construction depends on some knowledge on Hilbert C^*-modules (see Lance's book [11] for a good exposition).

$M_\iota = L^2(A_\iota, E_\iota)$ will denote the right Hilbert B-module obtained from A_ι by separation and completion with respect to the norm $\|a\| = \|\langle a, a\rangle_{M_\iota}\|^{1/2}$, where $\langle a_1, a_2\rangle_{M_\iota} = E_\iota(a_1^* a_2)$. Then the linear space $\mathcal{L}(M_\iota)$ of all adjointable B-module operators on M_ι is actually a C^*-algebra and we have a representation $\pi_\iota : A_\iota \to \mathcal{L}(M_\iota)$ defined by $\pi_\iota(a)\widehat{a'} = \widehat{aa'}$, where by $\hat{a}$ we denote the element of M_ι, corresponding to $a \in A_\iota$. π_ι is faithful by condition (1). Notice that $\pi_\iota|_B : B \to \mathcal{L}(M_\iota)$ makes M_ι a Hilbert $B - B$-bimodule. In this construction we have the specified element $\xi_\iota \stackrel{def}{=} \widehat{1_{A_\iota}} \in M_\iota$. We call the tripple $(\pi_\iota, M_\iota, \xi_\iota)$ the KSGNS representation of (A_ι, E_ι), i.e. $(\pi_\iota, M_\iota, \xi_\iota)$ =KSGNS(A_ι, E_ι) (KSGNS stands for Kasparov, Steinspring, Gel′fand, Naimark, Segal).

For every right B-module N one has operators $\theta_{x,y} \in \mathcal{L}(N)$ given by $\theta_{x,y}(n) = x\langle y, n\rangle_N$ $(x, y, n \in N)$. The C^*-subalgebra of $\mathcal{L}(N)$ that they generate is actually an ideal of $\mathcal{L}(N)$, which is denoted by $\mathcal{K}(N)$. It is an analogue of the C^*-algebra of all compact operators on a Hilbert space.

Since for every $\iota \in I$, $\theta_{\xi_\iota,\xi_\iota} \in \mathcal{L}(M_\iota)$ is the projection onto the Hilbert $B - B$-subbimodule $\xi_\iota B$ of M_ι it follows that $\xi_\iota B$ is a complemented submodule of M_ι. Therefore if $P_\iota^\circ = 1 - \theta_{\xi_\iota,\xi_\iota}$ then $\pi_\iota(b)P_\iota^\circ = P_\iota^\circ \pi_\iota(b) \in \mathcal{L}(M_\iota)$ for each $b \in B$. We define $M_\iota^\circ \stackrel{def}{=} P_\iota^\circ M_\iota$. If we view $\xi \stackrel{def}{=} 1_B$ as an element of the Hilbert $B - B$-bimodule

B, we can define

$$M = \xi B \oplus \bigoplus_{\substack{n\in\mathbb{N} \\ \iota_1,\dots,\iota_n\in I \\ \iota_1\neq\iota_2,\iota_2\neq\iota_3,\dots,\iota_{n-1}\neq\iota_n}} M^\circ_{\iota_1} \otimes_B M^\circ_{\iota_2} \otimes_B \cdots \otimes_B M^\circ_{\iota_n}, \tag{2}$$

where $\otimes_B$ means interior tensor product (see [11]). The Hilbert $B-B$-bimodule M constructed above is called the free product of $\{M_\iota, \iota \in I\}$ with respect to vectors $\{\xi_\iota, \iota \in I\}$ and is denoted by $(M, \xi) = \underset{\iota\in I}{*}(M_\iota, \xi_\iota)$.

For each $\iota \in I$ set

$$M(\iota) = \eta_\iota B \oplus \bigoplus_{\substack{n\in\mathbb{N} \\ \iota_1,\dots,\iota_n\in I \\ \iota_1\neq\iota_2,\iota_2\neq\iota_3,\dots,\iota_{n-1}\neq\iota_n \\ \iota_1\neq\iota}} M^\circ_{\iota_1} \otimes_B M^\circ_{\iota_2} \otimes_B \cdots \otimes_B M^\circ_{\iota_n}, \tag{3}$$

where $\eta_\iota \overset{def}{=} 1_B \in B$. We define a unitary operator

$$V_\iota : M_\iota \otimes_B M(\iota) \to M$$

given on elementary tensors by:

$$\begin{aligned}
&[\xi_\iota] \otimes [\eta_\iota] \mapsto \xi, \\
&[\zeta] \otimes [\eta_\iota] \mapsto \zeta, \text{ where } \zeta \in M^\circ_\iota \subset M \\
&[\xi_\iota] \otimes [\zeta_1 \otimes \cdots \otimes \zeta_n] \mapsto \zeta_1 \otimes \cdots \otimes \zeta_n, \text{ where } \zeta_j \in M^\circ_{\iota_j} \text{ and} \\
&\qquad\qquad \iota \neq \iota_1, \iota_1 \neq \iota_2, \dots, \iota_{n-1} \neq \iota_n \\
&[\zeta] \otimes [\zeta_1 \otimes \cdots \otimes \zeta_n] \mapsto \zeta \otimes \zeta_1 \otimes \cdots \otimes \zeta_n, \text{ where } \zeta \in M^\circ_\iota \text{ and} \\
&\qquad\qquad \zeta_j \in M^\circ_{\iota_j} \text{ with } \iota \neq \iota_1, \iota_1 \neq \iota_2, \dots, \iota_{n-1} \neq \iota_n.
\end{aligned}$$

Let $\lambda_\iota : A_\iota \to \mathcal{L}(M)$ be the $*$-homomorphism given by $\lambda_\iota(a) = V_\iota(\pi_\iota(a) \otimes 1)V_\iota^*$. λ_ι defines a left action of A_ι on M. Condition (1) implies that λ_ι is injective. Then A is defined as the C^*-subalgebra of $\mathcal{L}(M)$, generated by $\underset{\iota\in I}{\cup}\lambda_\iota(A_\iota)$, and $E : A \to B$ is the conditional expectation, given by $E(a) = \langle \xi, a(\xi)\rangle_M$. Note that if $b \in B$, then $\lambda_\iota(b) \in \mathcal{L}(M)$ does not depend on ι. $\lambda_\iota(b)$ gives the left action of B on M. Because of condition (1) for each $\iota \in I$ we have unital embeddings $A_\iota \hookrightarrow A$, which come from the $*$-homorphisms $\lambda_\iota : A_\iota \to \mathcal{L}(M)$. We will denote by π the representation $\pi : A \to \mathcal{L}(M)$ arising from the reduced amalgamated free product construction. We actually have that $(\pi, M, \xi) = \mathrm{KSGNS}(A, E)$.

Set $A^\circ_\iota = A_\iota \cap \ker(E_\iota)$. For $a \in A^\circ_\iota$, $\zeta_j \in M^\circ_{\iota_j}$ with $\iota_1, \dots, \iota_n \in I, n \geq 2$, and $\iota_j \neq \iota_{j+1}$ we have

$$\lambda_\iota(a)(\zeta_1 \otimes \cdots \otimes \zeta_n) = \begin{cases} \widehat{a} \otimes \zeta_1 \otimes \cdots \otimes \zeta_n, & \text{if } \iota \neq \iota_1, \\ (a(\zeta_1) - \xi_{\iota_1}\langle \xi_{\iota_1}, a(\zeta_1)\rangle) \otimes \zeta_2 \otimes \cdots \otimes \zeta_n + & \\ \qquad\qquad \pi_{\iota_2}(\langle \xi_{\iota_1}, a(\zeta_1)\rangle)\zeta_2 \otimes \cdots \otimes \zeta_n, & \text{if } \iota = \iota_1. \end{cases} \tag{4}$$

We will omit writing λ_ι and π_ι if this leads to no confusion.

We will use the following notation for the case of amalgamation which is similar to the notation in [4] used for the case of amalgamation over the scalars. If everything is as above by $\Lambda_B^\circ(\{A_\iota^\circ|\iota \in I\})$ we will denote the set of words of the form $a_1a_2\cdots a_n$, where $n \geq 1$ and $a_j \in A_{\iota_j}^\circ$ with $\iota_j \neq \iota_{j+1}$ for $1 \leq j \leq n-1$. We will not distinguish between two words from $\Lambda_B^\circ(\{A_\iota^\circ|\iota \in I\})$ which are equal as elements of A. We will denote $\Lambda_B(\{A_\iota^\circ|\iota \in I\}) \overset{def}{=} B \cup \Lambda_B^\circ(\{A_\iota^\circ|\iota \in I\})$. By $\mathbb{C}(A)$ we will denote the span of words from $\Lambda_B(\{A_\iota^\circ|\iota \in I\})$. Notice that $\mathbb{C}(A)$ is norm-dense in A. For a word $a_1a_2\cdots a_n \in \Lambda_B^\circ(\{A_\iota^\circ|\iota \in I\})$, where $n \geq 1$, $a_j \in A_{\iota_j}^\circ$ with $\iota_j \neq \iota_{j+1}$ for $1 \leq j \leq n-1$ we will consider to be of length n. Elements of B we will consider to be of length 0.

We will be mainly interested in the case $\mathrm{card}(I) = 2$ and that there exist states ϕ_ι on A_ι for $\iota = 1, 2$, such that these states are invariant under E_ι, i.e. for $\iota = 1, 2$ and $\forall a_\iota \in A_\iota$ we have $\phi_\iota(a_\iota) = \phi_\iota(E_\iota(a_\iota))$. We also require $\phi_1(b) = \phi_2(b)$ for $b \in B$. $\phi \overset{def}{=} \phi_B \circ E$, where $\phi_B \overset{def}{=} \phi_1|_B = \phi_2|_B$ is a well defined E-invariant state on $(A, E) = (A_1, E_1) * (A_2, E_2)$. In such case we will write formally

$$(A, E, \phi) = (A_1, E_1, \phi_1) * (A_2, E_2, \phi_2),$$

although the construction of (A, E) does not depend on ϕ_ι, $\iota = 1, 2$.

Let $Q_\iota : M \to M_\iota$ be the orthogonal projection of M onto the the complemented submodule $M_\iota = M_\iota^\circ \oplus B\widehat{1_A}$ (see (2). It is easy to see that $F_\iota : A \to A_\iota$ defined by $F_\iota(a) = \lambda_\iota(Q_\iota a Q_\iota^*)$ for $a \in A$ is a conditional expectation from A onto $A_\iota \subset A$ which is invariant with respect to E, i.e. $E(a) = E(F_\iota(a))$ for all $a \in A$.

We will need the following result concerning the faithfulness of E. This short proof was noted to me by Éric Ricard:

Theorem 2.1 (É. Ricard)**.** *Let $(A, E) = \underset{\iota \in I}{*}(A_\iota, E_\iota)$. Then the faithfulness of E_ι for $\forall \iota \in I$ implies the faithfulness of E.*

Proof. Let $K = \ker(E) \cap A^+$, then for any $x \in K$, $F_\iota(x) = 0$ for each $\iota \in I$ because of $0 = E(x) = E(F_\iota(x))$ and the faithfulness of $E_\iota = E|_{A_\iota}$. Then $x \in K$ implies $a_\iota^* x a_\iota \in K$ for each $a_\iota \in A_\iota$, since $E(a_\iota^* x a_\iota) = E(F_\iota(a_\iota^* x a_\iota)) = E(a_\iota^* F_\iota(x) a_\iota) = 0$.

Now if $x \in A$ is such that $x^*x \in K$, then for $y = xa_1\cdots a_n$ we have $y^*y \in K$, where $a_1a_2\cdots a_n \in \Lambda_B^\circ(\{A_\iota^\circ|\iota \in I\})$. In particular, it means that

$$0 = E(y^*y) = E((xa_1\cdots a_n)^* xa_1\cdots a_n) = \langle \widehat{1_A}, (xa_1\cdots a_n)^* xa_1\cdots a_n(\widehat{1_A})\rangle_M =$$

$$= \langle xa_1\cdots a_n(\widehat{1_A}), (xa_1\cdots a_n)^* xa_1\cdots a_n(\widehat{1_A})\rangle_M = \langle x(\widehat{a_1}\otimes\cdots\otimes\widehat{a_n}), x(\widehat{a_1}\otimes\cdots\otimes\widehat{a_n})\rangle_M$$

so x vanishes on the dense subset $\widehat{\mathbb{C}(A)}$ of M. Therefore $x \equiv 0$ as an operator of $\mathcal{L}(M)$. Thus $K = \{0\}$.

This proves the theorem. $\square$

It follows immediately from this theorem that if $(A, E, \phi) = (A_1, E_1, \phi_1)*(A_2, E_2, \phi_2)$ then the faithfulness of ϕ_1 and ϕ_2 imply the faithfulness of ϕ.

Now let's define the sets

$$\Lambda_B^1 \overset{def}{=} \mathrm{Span}\Big(\bigcup_{k=0}^{\infty} A_1^\circ (A_2^\circ A_1^\circ)^k\Big) \subset \mathbb{C}(A) \tag{5}$$

and

$$\Lambda_B^2 \overset{def}{=} \operatorname{Span}(\bigcup_{k=0}^{\infty} A_2^\circ(A_1^\circ A_2^\circ)^k) \subset \mathbb{C}(A). \tag{6}$$

Define also

$$\Lambda_B^{21} \overset{def}{=} \operatorname{Span}(\bigcup_{k=1}^{\infty} (A_2^\circ A_1^\circ)^k) \subset \mathbb{C}(A)$$

and

$$\Lambda_B^{12} \overset{def}{=} \operatorname{Span}(\bigcup_{k=1}^{\infty} (A_1^\circ A_2^\circ)^k) \subset \mathbb{C}(A).$$

Some of the most important examples are those of reduced C^*-algebras of amalgams of discrete groups. For each discrete group N we have the canonical tracial state $\tau_N \overset{def}{=} \langle \cdot, \widehat{1_H}\rangle_{l^2(H)}$ on $C_r^*(N)$. For each subgroup S of N we have a canonical conditional expectation $E_S^N : C_r^*(N) \to C_r^*(S)$ given on elements $\{\lambda_n, n \in N\}$ by

$$E_S^N(\lambda_n) = \begin{cases} \lambda_n, & \text{if } n \in S, \\ 0, & \text{if } n \notin S. \end{cases}$$

Let $G_1 \supset H \subset G_2$ be two discrete groups, containing a common subgroup (an isomorphic copy of H). Then we have $(C_r^*(G), E_H^G) = (C_r^*(G_1), E_H^{G_1}) * (C_r^*(G_2), E_H^{G_2})$, where $G = G_1 \underset{H}{*} G_2$.

The canonical tracial states $\tau_{G_\iota}, \iota = 1,2$ and τ_G are invariant under $E_H^{G_\iota}, \iota = 1,2$ and E_H^G respectivelly and $\tau_G = \tau_H \circ E_H^G$. Thus we can write formally

$$(C_r^*(G), E_H^G, \tau_G) = (C_r^*(G_1), E_H^{G_1}, \tau_{G_1}) * (C_r^*(G_2), E_H^{G_2}, \tau_{G_2}).$$

3. $\mathbf{K_0}^+$

We give the results of Germain and Pimsner first.

Theorem 3.1 ([7]). *Let $(A, \phi) = (A_1, \phi_1) * (A_2, \phi_2)$ is the reduced free product (with amalgamation over $\mathbb{C}$) of the unital, nuclear C^*-algebras A_1 and A_2 with respect to states ϕ_1 and ϕ_2. Then we have the following six term exact sequence:*

$$\begin{tikzcd} \mathbb{Z} \cong \mathbf{K_0}(\mathbb{C}) \arrow[r, "(\mathbf{K_0}(i_1), -\mathbf{K_0}(i_2))"] & \mathbf{K_0}(A_1) \oplus \mathbf{K_0}(A_2) \arrow[r, "\mathbf{K_0}(j_1)+\mathbf{K_0}(j_2)"] & \mathbf{K_0}(A) \arrow[d] \\ \mathbf{K_1}(A) \arrow[u] & \mathbf{K_1}(A_1) \oplus \mathbf{K_1}(A_2) \arrow[l, "\mathbf{K_1}(j_1)+\mathbf{K_1}(j_2)"'] & \mathbf{K_1}(\mathbb{C}) \cong 0, \arrow[l, "(\mathbf{K_1}(i_1), -\mathbf{K_1}(i_2))"'] \end{tikzcd}$$

where $i_k : \mathbb{C} \to A_k$ are the the unital $*$-homorphisms and $j_k : A_k \to A$ are the unital embeddings arising from the construction of reduced free product ($k = 1, 2$).

Theorem 3.2 ([13]). *Suppose that $G_1 \supset H \subset G_2$ are countable, discrete groups. Let $G = G_1 \underset{H}{*} G_2$. Then we have the following six term exact sequence:*

$$\begin{array}{ccccc}
\mathbf{K_0}(C_r^*(H)) & \xrightarrow{(\mathbf{K_0}(i_1),-\mathbf{K_0}(i_2))} & \mathbf{K_0}(C_r^*(G_1)) \oplus \mathbf{K_0}(C_r^*(G_2)) & \xrightarrow{\mathbf{K_0}(j_1)+\mathbf{K_0}(j_2)} & \mathbf{K_0}(C_r^*(G)) \\
\uparrow & & & & \downarrow \\
\mathbf{K_1}(C_r^*(G)) & \xleftarrow{\mathbf{K_1}(j_1)+\mathbf{K_1}(j_2)} & \mathbf{K_1}(C_r^*(G_1)) \oplus \mathbf{K_1}(C_r^*(G_2)) & \xleftarrow{(\mathbf{K_1}(i_1),-\mathbf{K_1}(i_2))} & \mathbf{K_1}(C_r^*(H)),
\end{array}$$

where $i_k : C_r^*(H) \to C_r^*(G_k)$ and $j_k : C_r^*(G_k) \to C_r^*(G)$ are the canonical inclusion maps ($k = 1, 2$).

Now suppose that we have unital C^*-algebaras $A_\iota,\ \iota = 1, 2$ and B. Suppose that we have unital inclusions $B \hookrightarrow A_\iota$ and conditional expectations $E_\iota : A_\iota \to B$ that satisfy property (1). Suppose also that for $\iota = 1, 2$ we have tracial states τ_ι on A_ι which satisfy $\tau_B \overset{def}{=} \tau_1|_B = \tau_2|_B$ and which are invariant under E_ι, i.e $\tau_\iota(a_\iota) = \tau_\iota(E_\iota(a_\iota))$ for each $a_\iota \in A_\iota$. Let us denote $(A, E, \tau) \overset{def}{=} (A_1, E_1, \tau_1) * (A_2, E_2, \tau_2)$ and let $j_\iota : A_\iota \to A$ are the inclusion maps, coming from the construction of reduced amalgamated free products. Suppose that $\tau \overset{def}{=} \tau_B \circ E$ is a faithful tracial state. Let's define

$$\Gamma \overset{def}{=} \mathbf{K_0}(j_1)(\mathbf{K_0}(A_1)) + \mathbf{K_0}(j_2)(\mathbf{K_0}(A_2)) \subset \mathbf{K_0}(A).$$

Then every element in Γ can be represented as

$$([p_1]_{\mathbf{K_0}(A)} - [q_1]_{\mathbf{K_0}(A)}) + ([p_2]_{\mathbf{K_0}(A)} - [q_2]_{\mathbf{K_0}(A)}),$$

where p_ι, q_ι are projections in some matrix algebras over A_ι for $\iota = 1, 2$. By expanding those matrices and adding zeros we can suppose without loss of generality that p_ι, q_ι are projections from $M_n(A_\iota)$ for some $n \in \mathbb{N}$ for $\iota = 1, 2$. Therefore every element of Γ can be represented in the form

$$\left[\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}\right]_{\mathbf{K_0}(A)} - \left[\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}\right]_{\mathbf{K_0}(A)}, \tag{7}$$

where now

$$\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \text{ and } \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \in M_{2n}(A).$$

We want to obtain a sufficient condition so that all elements $\gamma \in \Gamma$ for which $\mathbf{K_0}(\tau)(\gamma) > 0$ come from projections, i.e. $\exists m \in \mathbb{N}$ and $\rho \in M_m(A)$, such that $\gamma = [\rho]_{\mathbf{K_0}(A)}$ in $\mathbf{K_0}(A)$.

By definition the positive cone of $\mathbf{K_0}(A)$ is

$$\mathbf{K_0}(A)^+ = \{x \in \mathbf{K_0}(A) | \exists p \text{ projection in } M_n(A) \text{ for some } n \text{ with } x = [p]_{\mathbf{K_0}(A)}\}.$$

The scale of $\mathbf{K_0}(A)$ is

$$\Sigma(A) = \{x \in \mathbf{K_0}(A) | \exists p \text{ projection in } A \text{ with } x = [p]_{\mathbf{K_0}(A)}\}.$$

Dykema and Rørdam proved the following:

Theorem 3.3 ([6])**.** *Let $(A,\tau) = (A_1,\tau_1) * (A_2,\tau_2)$ be the reduced free product of the unital C^*-algebras A_1 and A_2 with respect to the faithful tracial states τ_1 and τ_2. Suppose that the Avitzour condition holds, namely there exist unitaries $u_1 \in A_1$ and $u_2, u_2' \in A_2$, such that $\tau_1(u_1) = \tau_2(u_2) = \tau_2(u_2') = \tau_2(u_1^* u_1') = 0$. Then we have*

$$\Gamma \cap \mathbf{K_0}(A)^+ = \{\gamma \in \Gamma | \mathbf{K_0}(\tau)(\gamma) > 0\} \cup \{0\}$$

and

$$\Gamma \cap \Sigma(A) = \{\gamma \in \Gamma | 0 < \mathbf{K_0}(\tau)(\gamma) < 1\} \cup \{0,1\}.$$

Notice that Theorem 3.1 implies that if A_1 and A_2 are nuclear then $\Gamma = \mathbf{K_0}(A)$.

Anderson, Blackadar and Haagerup proved this theorem for the case of $A = C_r^*(\mathbb{Z}_n * \mathbb{Z}_m)$ and gave one of the main technical tool for proving Theorem 3.3, which we will use here also:

Proposition 3.4 ([1])**.** *Let $\mathfrak{A}$ be a unital C^*-algebra and let ϕ be a faithful state on $\mathfrak{A}$. Suppose that $p, q \in \mathfrak{A}$ are projections that are ϕ-free in $\mathfrak{A}$. If $\phi(p) < \phi(q)$ then $\|p(1-q)\| < 1$ and there is a partial isometry $\nu \in \mathfrak{A}$ such that $\nu\nu^* = p$ and $\nu^*\nu < q$.*

Now we can state and prove our result:

Theorem 3.5. *Let A_ι be unital C^*-algebras that contain the unital C^*-algebra B as a unital C^*-subalgebra, i.e. $1_{A_\iota} \in B \subset A_\iota$, $\iota = 1,2$. Suppose that we have conditional expectations $E_\iota : A_\iota \to B$ and faithful tracial states τ_ι on A_ι for $\iota = 1,2$ such that $\tau_\iota = \tau_\iota \circ E_\iota$ and $\tau_1|_B = \tau_2|_B$. Form the reduced amalgamated free product $(A, E, \tau) = (A_1, E_1, \tau_1) * (A_2, E_2, \tau_2)$. Suppose that the following two conditions hold:*

(8)

$$\begin{cases} \forall b_1, \dots, b_l \in B, \text{ with } \tau(b_1) = \cdots = \tau(b_l) = 0, \ \exists m \in \mathbb{N} \text{ and unitaries} \\ \nu_{11}, \dots, \nu_{1m}, \nu_{21}, \dots, \nu_{2m} \text{ such that } \nu_{12}, \dots, \nu_{1m} \in A_1^\circ, \ \nu_{21}, \dots, \nu_{2(m-1)} \in A_2^\circ, \text{ and:} \\ \ \text{either } \nu_{11} \in A_1^\circ, \ \nu_{2m} \in A_2^\circ \text{ or} \\ \nu_{11} = 1_{A_1}, \ \nu_{2m} \in A_2^\circ, \text{ or} \\ \nu_{11} \in A_1^\circ, \ \nu_{2m} = 1_{A_2}, \\ \nu_{11} = 1_{A_1}, \ \nu_{2m} = 1_{A_2}, \ k \geq 2 \\ \ \text{with } E((\nu_{11}\nu_{21}\nu_{12}\cdots\nu_{1m}\nu_{2m}) b_k (\nu_{11}\nu_{21}\nu_{12}\cdots\nu_{1m}\nu_{2m})^*) = 0 \text{ for } k = 1, \dots, l, \\ (\text{i.e. there are unitaries that conjugate } B \ominus \mathbb{C}1_B \text{ out of } B) \end{cases}$$

and

$$\exists \text{ unitaries } u_1 \in A_1^\circ, \ u_2, u_2' \in A_2^\circ, \text{ with } E_2(u_2 u_2'^*) = 0. \tag{9}$$

Then:

$$\Gamma \cap \mathbf{K_0}(A)^+ = \{\gamma \in \Gamma | \mathbf{K_0}(\tau)(\gamma) > 0\} \cup \{0\}. \tag{10}$$

Proof. All elements of Γ have the form (7) for some $n \in \mathbb{N}$ and projections p_1, q_1 from $M_n(A_1)$ and p_2, q_2 from $M_n(A_2)$. Denote

$$\gamma = \left[\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \right]_{\mathbf{K_0}(A)} - \left[\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \right]_{\mathbf{K_0}(A)}.$$

Consider

$$P \stackrel{def}{=} \begin{pmatrix} U_2 & 0 \\ 0 & U_2U_1 \end{pmatrix} \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \begin{pmatrix} U_2^* & 0 \\ 0 & U_1^*U_2^* \end{pmatrix} \text{ and}$$

$$Q \stackrel{def}{=} \begin{pmatrix} U_2 & 0 \\ 0 & U_2U_1 \end{pmatrix} \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \begin{pmatrix} U_2^* & 0 \\ 0 & U_1^*U_2^* \end{pmatrix},$$

where $U_1 = \mathrm{diag}(u_1, \dots, u_1) \in M_n(A_1)$ and $U_2 = \mathrm{diag}(u_2, \dots, u_2) \in M_n(A_2)$. It is clear that $P, Q \in M_{2n}(\Lambda_B^2 \oplus B1_B)$. For $T \in M_m(A)$ we will denote by T_{ij} the ij-entry of T. Now consider the set of elements $S_P = \{E(P_{ij}) - \tau(P_{ij}) | 1 \le i, j \le 2n\} \cup \{E(u_1P_{ij}u_1^*) - \tau(u_1P_{ij}u_1^*) | 1 \le i, j \le 2n\}$ and the set $S_Q = \{E(Q_{ij}) - \tau(Q_{ij}) | 1 \le i, j \le 2n\} \cup \{E(u_1Q_{ij}u_1^*) - \tau(u_1Q_{ij}u_1^*) | 1 \le i, j \le 2n\}$.

Applying condition (8) to the set S_P we obtain unitaries $\nu_{ij}, i = 1, 2, j = 1, \dots, m_P$. Set

$$W_P \stackrel{def}{=} \begin{cases} \nu_{11}\nu_{21}\nu_{12} \cdots \nu_{2(m_P-1)}\nu_{1m_P}, & \text{if } \nu_{2m_P} = 1_{A_2},\ \nu_{11} \in A_1^\circ, \\ \nu_{11}\nu_{21}\nu_{12} \cdots \nu_{2(m_P-1)}\nu_{1m_P}\nu_{2m_P}u_1, & \text{if } \nu_{2m_P} \in A_2^\circ,\ \nu_{11} \in A_1^\circ, \\ u_1\nu_{21}\nu_{12} \cdots \nu_{2(m_P-1)}\nu_{1m_P}\nu_{2m_P}u_1, & \text{if } \nu_{2m_P} \in A_2^\circ,\ \nu_{11} = 1_{A_1}. \\ u_1\nu_{21}\nu_{12} \cdots \nu_{2(m_P-1)}\nu_{1m_P}, & \text{if } \nu_{2m_P} = 1_{A_2},\ \nu_{11} = 1_{A_1},\ k \ge 2. \end{cases}$$

Applying condition (8) to the set S_Q we obtain unitaries $\nu'_{ij}, i = 1, 2, j = 1, \dots, m_Q$. Set

$$W_Q \stackrel{def}{=} \begin{cases} \nu'_{11}\nu'_{21}\nu'_{12} \cdots \nu'_{2(m_P-1)}\nu'_{1m_P}, & \text{if } \nu'_{2m_P} = 1_{A_2},\ \nu'_{11} \in A_1^\circ, \\ \nu'_{11}\nu'_{21}\nu'_{12} \cdots \nu'_{2(m_P-1)}\nu'_{1m_P}\nu'_{2m_P}u_1, & \text{if } \nu'_{2m_P} \in A_2^\circ,\ \nu'_{11} \in A_1^\circ, \\ u_1\nu'_{21}\nu'_{12} \cdots \nu'_{2(m_P-1)}\nu'_{1m_P}\nu'_{2m_P}u_1, & \text{if } \nu'_{2m_P} \in A_2^\circ,\ \nu'_{11} = 1_{A_1}. \\ u_1\nu'_{21}\nu'_{12} \cdots \nu'_{2(m_P-1)}\nu'_{1m_P}, & \text{if } \nu'_{2m_P} = 1_{A_2},\ \nu'_{11} = 1_{A_1},\ k \ge 2. \end{cases}$$

It is easy to see that $W_PPW_P^*,\ W_QQW_Q^* \in M_{2n}(\Lambda_B^1 \oplus \mathbb{C}1_B)$.

Now consider the following matrix in $M_{2n}(A)$:

$$U = \left(\frac{\omega^{ij}}{\sqrt{2n}} u'_2(u_1u_2)^{2ni+j}u_2'^*\right)_{i,j=1}^{2n},$$

where $\omega = \exp(2\pi\sqrt{-1}/2n)$ is a primitive $2n$-th root of 1. It is clear that $U \in M_{2n}(\Lambda_B^2)$. We will check that U is a unitary matrix:

$$(UU^*)_{ij} = (2n)^{-1}\sum_{k=1}^{2n}\omega^{ik}u'_2(u_1u_2)^{2ni+k}\omega^{-jk}(u_1u_2)^{-2nj-k}u_2'^* =$$

$$(2n)^{-1}\sum_{k=1}^{2n}\omega^{(i-j)k}u'_2(u_1u_2)^{2n(i-j)}u_2'^* = (2n)^{-1}u'_2(u_1u_2)^{2n(i-j)}u_2'^*\sum_{k=1}^{2n}\omega^{(i-j)k} = \delta_{ij}1_A.$$

$$(U^*U)_{ij} = (2n)^{-1}\sum_{k=1}^{2n}\omega^{-ik}u'_2(u_1u_2)^{-2nk-i}\omega^{jk}(u_1u_2)^{2nk+j}u_2'^* =$$

$$(2n)^{-1}\sum_{k=1}^{2n}\omega^{(j-i)k}u'_2(u_1u_2)^{j-i}u_2'^* = (2n)^{-1}u'_2(u_1u_2)^{j-i}u_2'^*\sum_{k=1}^{2n}\omega^{(j-i)k} = \delta_{ij}1_A.$$

Thus $U \in M_{2n}(A)$ is a unitary.

Take $T \in M_{2n}(\Lambda_B^1 \oplus \mathbb{C}1_B)$. Then $T = T_0 + T_1 \otimes 1_A$, with $T_0 \in M_{2n}(\Lambda_B^1)$ and $T_1 \in M_{2n}(\mathbb{C})$. It is easy to see that $UT_0U^* \in M_{2n}(\Lambda_B^2)$. Now if $T_1 = (t_{ij})_{i,j=1}^{2n}$ then for $U(T_1 \otimes 1_A)U^* = (s_{ij})_{i,j=1}^{2n}$ we have

$$s_{ij} = (2n)^{-1}\sum_{k=1}^{2n}\sum_{l=1}^{2n}\omega^{ik}u_2'(u_1u_2)^{2ni+k}u_2'^*t_{kl}\omega^{-jl}u_2'(u_1u_2)^{-2nj-l}u_2'^* =$$

$$(2n)^{-1}\sum_{k=1}^{2n}\sum_{l=1}^{2n}t_{kl}\omega^{ik-jl}u_2'(u_1u_2)^{2ni+k-2nj-l}u_2'^*.$$

If $i \neq j$ then $2ni + k - 2nj - l \neq 0$ for any $1 \leq k, l \leq 2n$, so in this case $s_{ij} \in \Lambda_B^2$. If $i = j$ then:

$$s_{ii} = (2n)^{-1}\sum_{k=1}^{2n}\sum_{l=1}^{2n}t_{kl}\omega^{i(k-l)}u_2'(u_1u_2)^{k-l}u_2'^* =$$

$$(2n)^{-1}\sum_{\substack{1\leq k,l\leq 2n\\ k\neq l}}t_{kl}\omega^{i(k-l)}u_2'(u_1u_2)^{k-l}u_2'^* + ((2n)^{-1}\sum_{k=1}^{2n}t_{kk}) \otimes 1_A.$$

So $s_{ii} = s_{ii}' + \mathrm{tr}_{2n}(T_1) \otimes 1_A$, where $s_{ii}' \in \Lambda_B^2$. All this means that $U(T_1 \otimes 1_A)U^* = T_1' + \mathrm{tr}_{2n}(T_1)1_A \otimes 1_{M_{2n}(\mathbb{C})}$, with $T_1' \in M_{2n}(\Lambda_B^2)$, which implies that $UTU^* \in M_{2n}(\Lambda_B^2) \oplus \mathbb{C}1_{M_{2n}(A)}$.

This means that we have

$$P' \overset{def}{=} UW_PPW_P^*U^* \in M_{2n}(\Lambda_B^2) \oplus \mathbb{C}1_{M_{2n}(A)} \tag{11}$$

and

$$Q' \overset{def}{=} u_1UW_QQW_Q^*U^*u_1^* \in M_{2n}(\Lambda_B^1) \oplus \mathbb{C}1_{M_{2n}(A)}. \tag{12}$$

It is clear that $\mathrm{tr}_{2n}\otimes E(P') = \mathrm{tr}_{2n}\otimes\tau(P')$ and that $\mathrm{tr}_{2n}\otimes E(Q') = \mathrm{tr}_{2n}\otimes\tau(Q')$. Since P' and Q' are nontrivial projections it is also clear that $C^*(\{P', 1_A\})$ and $C^*(\{Q', 1_A\})$ are both 2-dimensional. Therefore for any $p \in C^*(\{P', 1_A\})$ and $q \in C^*(\{Q', 1_A\})$ we have $\mathrm{tr}_{2n} \otimes E(p) = \mathrm{tr}_{2n} \otimes \tau(p)$ and $\mathrm{tr}_{2n} \otimes E(q) = \mathrm{tr}_{2n} \otimes \tau(q)$. Therefore from (11), (12) and the definition of freeness it follows that P' is both $\mathrm{tr}_{2n} \otimes E$-free and $\mathrm{tr}_{2n} \otimes \tau$-free from Q'.

Since $\mathrm{tr}_{2n}\otimes\tau$ is a faithful tracial state (because of faithfulness of τ_1, τ_2 and Theorem 2.1) and because

$$\mathrm{tr}_{2n} \otimes \tau(P') = (2n)^{-1}\mathbf{K_0}(\tau)(P) > (2n)^{-1}\mathbf{K_0}(\tau)(Q) = \mathrm{tr}_{2n} \otimes \tau(Q'),$$

we can apply Proposition 3.4 and conclude that there is a projection $Q'' < P'$ and a partial isometry ν with $\nu\nu^* = Q'$ and $\nu^*\nu = Q''$. Thus $\gamma = [P' - Q'']_{\mathbf{K_0}(A)}$ in $\mathbf{K_0}(A)$. This proves the theorem. $\square$

Corollary 3.6. *Suppose that $G_1 \supsetneq H \subsetneq G_2$ are countable discrete groups with $H \neq \{1\}$. Suppose that $\exists g \in G \stackrel{def}{=} G_1 \underset{H}{*} G_2$ with $g(H\backslash\{1\})g^{-1} \cap H = \emptyset$. Suppose also that $\mathbf{K_1}(C_r^*(H)) = 0$. Then*

$$\mathbf{K_0}(C_r^*(G))^+ = \{\gamma \in \mathbf{K_0}(C_r^*(G)) | \mathbf{K_0}(\tau_G)(\gamma) > 0\} \cup \{0\}.$$

Proof. Because of the existence of γ we see that condition (8) of Theorem 3.5 is satisfied. The existence of γ implies also that H is not normal in at least one of the groups G_1 or G_2. Suppose without loss of generality that H is not normal in G_2. Then $\mathrm{Index}[G_1 : H] \geq 2$ and $\mathrm{Index}[G_2 : H] \geq 3$ so we can find $g_1 \in G_1\backslash H$ and $g_2, g_2' \in G_2\backslash H$ with $g_2 g_2'^{-1} \in G_2\backslash H$. Then condition (9) is satisfied with elements $u_1 = \lambda_{g_1}$, $u_2 = \lambda_{g_2}$ and $u_2' = \lambda_{g_2'}$ and therefore we can apply Theorem 3.5. From the fact that $\mathbf{K_1}(C_r^*(H)) = 0$ and Theorem 3.2 it follows that $\Gamma = \mathbf{K_0}(C_r^*(G))$. This proves the corollary. $\square$

Remark 3.7. *Condition (9) is an analogue of the Avitzour condition for the case of reduced amalgamated free products. We will use it in the next section to prove simplicity and uniqueness of trace.*

4. Simplicity and Uniqueness of Trace

In this section we will use Power's idea ([14]) to obtain a sufficient condition for simplicity and uniqueness of trace for reduced amalgamated free product C^*-algebras. We will make use the following result (due to Avitzour) and its proof:

Theorem 4.1 ([2])**.** *Let A_1 and A_2 be two unital C^*-algebras and ϕ_1 respectivelly ϕ_2 states on them with faithfil GNS-representations. Suppose that there are unitaries $u_i \in A_i$, $i = 1, 2$ such that ϕ_1 and ϕ_2 are invariant with respect to conjugation by u_1 and u_2 respectivelly and such that $\phi_i(u_i) = 0$ for $i = 1, 2$. Suppose also that there is a unitary $u_2' \in A_2$, such that $\phi_2(u_2') = 0$ and $\phi_2(u_2^* u_2') = 0$. Then:*

(I) $(A, \phi) \stackrel{def}{=} (A_1, \phi_1) * (A_2, \phi_2)$ is simple.

(II) If ϕ is invariant with respect to conjugation by u_2' then ϕ is the only state on A which is invariant with respect to conjugation by u_1, u_2, u_2'. If ϕ is not invariant with respect to conjugation by u_2' then there is no state on A which is invariant with respect to conjugation by u_1, u_2, u_2'.

The proof of Theorem 4.1 uses a lemma of Choi from [3]. We will need the following straightforward generalization of this lemma to the case of Hilbert modules:

Lemma 4.2. *Let H_1 and H_2 be right Hilbert B-modules. Let $u_1, \dots, u_n \in \mathcal{L}(H_1 \oplus H_2)$ be unitaries such that $u_i^* u_j(H_2) \perp H_2$, whenever $i \neq j$. Suppost that $b \in \mathcal{L}(H_1 \oplus H_2)$ is such that $b(H_1) \perp H_1$. Then $\|\frac{1}{n}\sum_{k=1}^{n} u_i^* b u_i\| \leq 2\|b\|/\sqrt{n}$.*

Proof. First assume that

$$b = \begin{bmatrix} 0 & 0 \\ b_1 & b_2 \end{bmatrix} \in \mathcal{L}(H_1 \oplus H_2).$$

If
$$c = \begin{bmatrix} c_1 & c_2 \\ 0 & 0 \end{bmatrix} \in \mathcal{L}(H_1 \oplus H_2)$$
then for $x \oplus y \in H_1 \oplus H_2$ we have
$$\begin{bmatrix} c_1 & c_2 \\ b_1 & b_2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1x + c_2y \\ b_1x + b_2y \end{pmatrix}.$$

Then:
$$\left\| \begin{pmatrix} c_1x + c_2y \\ b_1x + b_2y \end{pmatrix} \right\|_B^2 = \|\langle (c_1x + c_2y) \oplus (b_1x + b_2y), (c_1x + c_2y) \oplus (b_1x + b_2y) \rangle_{H_1 \oplus H_2}\|_B =$$
$$= \|\langle c_1x + c_2y, c_1x + c_2y \rangle_{H_1} + \langle b_1x + b_2y, b_1x + b_2y \rangle_{H_2}\|_B \leq$$
$$\|\langle c_1x + c_2y, c_1x + c_2y \rangle_{H_1}\|_B + \|\langle b_1x + b_2y, b_1x + b_2y \rangle_{H_2}\|_B = \|c_1x + c_2y\|_B^2 + \|b_1x + b_2y\|_B^2$$
$$= \left\| \begin{bmatrix} c_1 & c_2 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|_B^2 + \left\| \begin{bmatrix} 0 & 0 \\ b_1 & b_2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\|_B^2.$$

Taking supremum on both sides over all vectors $x \oplus y$ in the unit ball of $H_1 \oplus H_2$ we get
$$\left\| \begin{bmatrix} c_1 & c_2 \\ b_1 & b_2 \end{bmatrix} \right\|^2 = \|c + b\|^2 \leq \|c\|^2 + \|b\|^2.$$

Now $u_j^* u_i b u_i^* u_j(H_2) \subseteq u_j u_i^* b(H_1) = 0$. So $u_j^* u_i b u_i^* u_j$ has the form $\begin{bmatrix} c_1 & c_2 \\ 0 & 0 \end{bmatrix}$. Now $\|\sum\limits_{i=1}^{n} u_i b u_i^*\|^2 = \|u_1^*(\sum\limits_{i=1}^{n} u_i b u_i^*) u_1\|^2 = \|b + \sum\limits_{i=2}^{n} u_1^* u_i b u_i^* u_1\| \leq \|b\|^2 + \|\sum\limits_{i=2}^{n} u_1^* u_i b u_i^* u_1\|^2 = \|b\|^2 + \|\sum\limits_{i=2}^{n} u_i b u_i^*\|^2$. It follows by induction that $\|\sum\limits_{i=1}^{n} u_i b u_i^*\|^2 \leq n\|b\|^2$. For the general case we represent
$$b = \begin{bmatrix} 0 & b_3 \\ b_1 & b_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ b_1 & b_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ b_3^* & 0 \end{bmatrix}^*.$$

Then
$$\|\sum_{i=1}^{n} u_i b u_i^*\| \leq \|\sum_{i=1}^{n} u_i \begin{bmatrix} 0 & 0 \\ b_1 & b_2 \end{bmatrix} u_i^*\| + \|\sum_{i=1}^{n} u_i \begin{bmatrix} 0 & 0 \\ b_3^* & 0 \end{bmatrix} u_i^*\| \leq$$
$$\sqrt{n} \left\| \begin{bmatrix} 0 & 0 \\ b_1 & b_2 \end{bmatrix} \right\| + \sqrt{n} \left\| \begin{bmatrix} 0 & 0 \\ b_3^* & 0 \end{bmatrix} \right\| \leq 2\sqrt{n}\|b\|.$$

□

Untill the end of the section we will assume that we have unital C^*-algebras A_1, A_2 that contain the unital C^*-algebra B as a unital C^*-subalgebra. We will also assume that we have condiditonal expectations $E_i : A_i \to B$ for $i = 1, 2$ that have faithful KSGNS-representations (i.e. satisfy condition (1)). We now form the reduced amalgamated free product $(A, E) \overset{def}{=} (A_1, E_1) * (A_2, E_2)$.

Now we can imitate Avitzour's proof of Theorem 4.1 and prove the following version for the amalgamated case:

Proposition 4.3. *Suppose everything is as above and also suppose that there are unitaries $u_1 \in A_1$, $u_2, u_2' \in A_2$ with $E_1(u_1) = 0 = E_2(u_2) = E_2(u_2') = E(u_2 u_2'^*)$. Then if $x \in \Lambda_B^1$ then $0 \in \overline{\text{conv}}\{uxu^* | u \in A$ is a unitary $\}$.*

Proof. We will use the notation from section 2 with $I = \{1, 2\}$. Let $W_0 \subset \mathbb{C}(A)$ be the span of all words from $\Lambda_B(A_1^\circ, A_2^\circ)$ that either begin with an element $a_1 \in A_1^\circ$ or begin with $u_2^* b$ with $b \in B$, or come from B. Let $W_1 \subset \mathbb{C}(A)$ be the span of all words from $\Lambda_B(A_1^\circ, A_2^\circ)$ that begin with an element $a_2 \in A_2^\circ$ satisfying $E_2(u_2 a_2) = 0$. Denote

$$H_i \stackrel{def}{=} \overline{\pi(W_i)\widehat{1_A}} \subset M, \ i = 0, 1$$

We have $M = H_0 \oplus H_1$ as right Hilbert B-module (the orthogonality is with respect to $\langle ., . \rangle_M$). To show this notice first that $\text{Span}(W_0 \cup W_1)$ is dense in A. Therefore $M = H_0 + H_1$. For every word $w_0 \in W_0$ and every word $w_1 \in W_1$we have $E(w_0^* w_1) = 0$ which is easy to see by considering the three possible cases for w_0. Thus $H_0 \perp H_1$ by linearity.

We claim that $(u_2^* u_1)^k (H_1) \subseteq H_0$ for $k \neq 0$.

It is enough to prove that $(u_2^* u_1)^k W_1 \subseteq W_0$.

If $k > 0$ then $(u_2^* u_1)^k W_1$ is spanned by words from $\Lambda_B^\circ(A_1^\circ, A_2^\circ)$ starting with u_2^*. If $k < 0$ then take any word $w_1 \in W_1$. Then $w_1 = a_2 w_1'$,where $a_2 \in A_2^\circ$ satisfies $E(u_2 a_2) = 0$ and $w_1' \in \Lambda_B^\circ(A_1^\circ, A_2^\circ)$ starts with an element of A_1°. Then

$$(u_2^* u_1)^k w_1 = (u_1^* u_2)^{-k} a_2 w_1' = (u_1^* u_2)^{-k-1} u_1^* (u_2 a_2) w_1'$$

is a word, starting with $u_1^* \in A_1^\circ$. Thus $(u_2^* u_1)^k W_1 \subseteq W_0$.

Now $u_2'^* x u_2' \in \Lambda_B^2$ and also it is clear that $(u_2'^* x u_2')(W_0) \subseteq W_1$ by considering the three possibilities for W_0 (notice that $E(u_2' u_2^* b) = 0 \ \forall b \in B$). Now we can use Lemma 4.2 and get

$$\| \frac{1}{N} \sum_{k=1}^{N} (u_2'^* u_1)^k (u_2'^* x u_2') (u_2'^* u_1)^{-k} \| \leq \frac{2\|x\|}{\sqrt{N}}.$$

This implies that $0 \in \overline{\text{conv}}\{uxu^* | u \in A$ is a unitary $\}$. □

We will prove the next technical lemma:

Lemma 4.4. *Suppose that everything is as above and suppose that there are states ϕ_i on A_i for $i = 1, 2$ which are invariant with respect to E_i, $i = 1, 2$ and satisfy $\phi_1|_B = \phi_2|_B (\stackrel{def}{=} \phi_B)$, and construct $\phi \stackrel{def}{=} \phi_B \circ E$.*

Suppose that there are two multiplicative sets $1_A \in \tilde{A}_i \subset A_i$ such that $\text{Span}(\tilde{A}_i)$ is dense in A_i, suppose from $a_i \in \tilde{A}_i$ follows $E_i(a_i)$, $a_i - E_i(a_i)$, $a_i - \phi_i(a_i) \in \tilde{A}_i$, for $i = 1, 2$, and $B \cap \tilde{A}_1 = B \cap \tilde{A}_2 \stackrel{def}{=} \tilde{B}$.

Suppose also that there are two sets of unitaries $\emptyset \neq W_i \subset \tilde{A}_i \cap A_i^\circ$ such that $(W_i)^* \subset \tilde{A}_i$ for $i = 1, 2$. Let $u_i \in W_i$, $i = 1, 2$ and suppose that ϕ is invariant with respect to conjugation by u_1 and u_2.

Suppose also that the following condition, similar to condition (8), holds:
(13)

$$\begin{cases} \forall b_1,\dots,b_l\in\tilde{B},\ \textit{with } \phi(b_1)=\cdots=\phi(b_l)=0,\ \exists m\in\mathbb{N} \textit{ and unitaries}\\ \nu_{11},\dots,\nu_{1m},\nu_{21},\dots,\nu_{2m} \textit{ such that } \nu_{12},\dots,\nu_{1m}\in W_1,\ \nu_{21},\dots,\nu_{2(m-1)}\in W_2,\ \textit{and:}\\ \textit{either } \nu_{11}\in W_1,\ \nu_{2m}\in W_2 \textit{ or}\\ \nu_{11}=1_{A_1},\ \nu_{2m}\in W_2,\ \textit{or}\\ \nu_{11}\in W_1,\ \nu_{2m}=1_{A_2},\\ \nu_{11}=1_{A_1},\ \nu_{2m}=1_{A_2},\ k\geq 2\\ \ \textit{with } E((\nu_{11}\nu_{21}\nu_{12}\cdots\nu_{1m}\nu_{2m})b_k(\nu_{11}\nu_{21}\nu_{12}\cdots\nu_{1m}\nu_{2m})^*)=0 \textit{ for } k=1,\dots,l,\\ (\textit{i.e. there are unitaries that conjugate } \tilde{B}\ominus\mathbb{C}1_B \textit{ out of } B) \end{cases}$$

Suppose finally that there are unitaries $\omega_1\in W_1$ and ω_2 with $\omega_2=1_A$ or $\omega_2\in W_2$, such that $\forall b\in\tilde{B},\ \exists\omega_1^b\in W_1,$ and $\omega_2^b\in W_2$ if $\omega_2\in W_2$ or $\omega_2^b=1$ if $\omega_2=1$ with $E((\omega_2^b)^*(\omega_1^b)^*b\omega_1\omega_2)=0$.

Then given $x\in\mathrm{Alg}(\tilde{A}_1\cup\tilde{A}_2)$ with $\phi(x)=0$ there exist unitaries $\alpha_1,\dots,\alpha_s$ with $\alpha_i\in W_{1+(i \mod 2)}$ such that $\alpha_1^*\cdots\alpha_s^*x\alpha_s\cdots\alpha_1\in\Lambda_B^2$.

Proof. Until the end of this proof we will use the following sets:

$$\tilde{\Lambda}_B^1\overset{def}{=}\ \mathrm{Span}(\bigcup_{k=0}^{\infty}(A_1^\circ\cap\tilde{A}_1)\cdot[(A_2^\circ\cap\tilde{A}_2)\cdot(A_1^\circ\cap\tilde{A}_1)]^k\subset\mathbb{C}(A),$$

$$\tilde{\Lambda}_B^2\overset{def}{=}\ \mathrm{Span}(\bigcup_{k=0}^{\infty}(A_2^\circ\cap\tilde{A}_2)\cdot[(A_1^\circ\cap\tilde{A}_1)\cdot(A_2^\circ\cap\tilde{A}_2)]^k)\subset\mathbb{C}(A),$$

$$\tilde{\Lambda}_B^{21}\overset{def}{=}\ \mathrm{Span}(\bigcup_{k=1}^{\infty}[(A_2^\circ\cap\tilde{A}_2)\cdot(A_1^\circ\cap\tilde{A}_1)]^k)\subset\mathbb{C}(A),$$

$$\tilde{\Lambda}_B^{12}\overset{def}{=}\ \mathrm{Span}(\bigcup_{k=1}^{\infty}[(A_1^\circ\cap\tilde{A}_1)\cdot(A_2^\circ\cap\tilde{A}_2)]^k)\subset\mathbb{C}(A).$$

We can write $x=x_B+x_1+x_2+x_{12}+x_{21}$, where $x_B\in\mathrm{Span}(\tilde{B})$ with $\phi(x_B)=0$, $x_1\in\tilde{\Lambda}_B^1$, $x_2\in\tilde{\Lambda}_B^2$, $x_{12}\in\tilde{\Lambda}_B^{12}$ and $x_{21}\in\tilde{\Lambda}_B^{21}$. We will be alternativelly conjugating x with unitaries from W_1 and W_2 until we end up with an element of $\tilde{\Lambda}_B^2$. So at the start we call the words from $\tilde{\Lambda}_B^1$ "good words". When we conjugate a word $w_1\in\tilde{\Lambda}_B^1$ with $a_2\in W_2$ we end up with a word $a_2w_1a_2^*\in\tilde{\Lambda}_B^2$. Now we call the words of $\tilde{\Lambda}_B^2$ "good words". If we now take a word $w_2\in\tilde{\Lambda}_B^2$ and conjugate it with an element $a_1\in W_1$ we obtain the word $a_1w_2a_1^*\in\tilde{\Lambda}_B^1$ so we can call the words from $\tilde{\Lambda}_B^1$ "good words". We will show that proceeding in this way, i.e. alternativelly conjugating x with elements from W_1 and W_2 we can come to an element $\alpha_1^*\cdots\alpha_s^*x\alpha_s\cdots\alpha_1\in\tilde{\Lambda}_B^2$ consisting of a linear combination of "good words" from $\tilde{\Lambda}_B^2$. This will prove the lemma.

We have to consider the following 4 possibilies:

(i) Take a word $b \in \tilde{B}$. Suppose that the "good words" are in $\tilde{\Lambda}_B^2$ and we are going to conjugate b with the element $u_1 \in W_1$. Then we obtain

$$u_1 b u_1^* = E(u_1 b u_1^*) + (u_1 b u_1^* - E(u_1 b u_1^*))$$

for which $(u_1 b u_1^* - E(u_1 b u_1^*)) \in \tilde{A}_1 \cap A_1^\circ \subset \tilde{\Lambda}_1$ is a "good word" and the word $E(u_1 b u_1^*) \in \tilde{B}$ satisfies $\phi(E(u_1 b u_1^*)) = \phi(b)$. Analoguous conclusion can be drawn if we suppose that the "good words" are in $\tilde{\Lambda}_B^1$ and we are conjugating with the element $u_2 \in W_2$.

(ii) Take a word $\gamma_1 \cdots \gamma_{2n} \in \tilde{\Lambda}_B^{12}$ ($\gamma_i \in A^\circ_{1+(i-1 \mod 2)} \cap \tilde{A}_{1+(i-1 \mod 2)}$) and conjugate it with a unitary $a_2 \in W_2$ thinking that the "good words" are in $\tilde{\Lambda}_B^1$. We get

$$a_2\gamma_1 \cdots \gamma_{2n-1}\gamma_{2n}a_2^* = a_2\gamma_1 \cdots \gamma_{2n-1}E(\gamma_{2n}a_2^*) + a_2\gamma_1 \cdots \gamma_{2n-1}(\gamma_{2n}a_2^* - E(\gamma_{2n}a_2^*)).$$

The first word is from $\tilde{\Lambda}_B^{21}$ of the same length $2n$ as the word $\gamma_1 \cdots \gamma_{2n-1}\gamma_{2n}$ and the second word is from $\tilde{\Lambda}_B^2$, i.e. a "good word". If we supposed that the good words were in $\tilde{\Lambda}_B^2$ and we were conjugating with a unitary $a_1 \in W_1$ then we would have

$$a_1\gamma_1 \cdots \gamma_{2n-1}\gamma_{2n}a_2^* = E(a_1\gamma_1)\gamma_2 \cdots \gamma_{2n-1}\gamma_{2n}a_1^* + (a_1\gamma_1 - E(a_1\gamma_1))\gamma_2 \cdots \gamma_{2n-1}\gamma_{2n}a_1^*$$

So again we end up with a word from $\tilde{\Lambda}_B^{21}$ of length $2n$ and a "good word" from $\tilde{\Lambda}_B^1$.

(iii) In a similar way we can treat a word $\gamma_2 \cdots \gamma_{2n+1} \in \tilde{\Lambda}_B^{21}$ ($\gamma_i \in A^\circ_{1+(i-1 \mod 2)} \cap \tilde{A}_{1+(i-1 \mod 2)}$). If we conjugate with a unitary $a_2 \in W_2$ knowing that the "good words" are in $\tilde{\Lambda}_B^1$ we end up with

$$a_2\gamma_2\gamma_3 \cdots \gamma_{2n+1}a_2^* = E(a_2\gamma_2)\gamma_3 \cdots \gamma_{2n+1}a_2^* + (a_2\gamma_2 - E(a_2\gamma_2))\gamma_3 \cdots \gamma_{2n+1}a_2^*.$$

The first word is from $\tilde{\Lambda}_B^{12}$ and of the same length $2n$ and the second word is from $\tilde{\Lambda}_B^2$, i.e. a "good word". In the same way if the good words were in $\tilde{\Lambda}_B^2$ and we were conjugating with a unitary $a_1 \in W_1$ we would obtain

$$a_1\gamma_2 \cdots \gamma_{2n}\gamma_{2n+1}a_1^* = a_1\gamma_2 \cdots \gamma_{2n}E(\gamma_{2n+1}a_1^*) + a_1\gamma_2 \cdots \gamma_{2n}(\gamma_{2n+1}a_1^* - E(\gamma_{2n+1}a_1^*)).$$

The first word is from $\tilde{\Lambda}_B^{12}$ of length $2n$ and the second word is from $\tilde{\Lambda}_B^1$, i.e. a "good word".

(iv) Take a word $\gamma_2 \cdots \gamma_{2n} \in \tilde{\Lambda}_B^2$ ($\gamma_i \in A^\circ_{1+(i-1 \mod 2)} \cap \tilde{A}_{1+(i-1 \mod 2)}$). If the "good words" are in $\tilde{\Lambda}_B^1$ and if we conjugate this word with the unitary $u_2 \in W_2$, we will get

$$\begin{aligned} u_2\gamma_2\gamma_3 \cdots \gamma_{2n-1}\gamma_{2n}u_2^* &= E(u_2\gamma_2)\gamma_3 \cdots \gamma_{2n-1}E(\gamma_{2n}u_2^*) + \\ &+ (u_2\gamma_2 - E(u_2\gamma_2))\gamma_3 \cdots \gamma_{2n-1}E(\gamma_{2n}u_2^*) + E(u_2\gamma_2)\gamma_3 \cdots \gamma_{2n-1}(\gamma_{2n}u_2^* - E(\gamma_{2n}u_2^*)) + \\ &+ (u_2\gamma_2 - E(u_2\gamma_2))\gamma_3 \cdots \gamma_{2n-1}(\gamma_{2n}u_2^* - E(\gamma_{2n}u_2^*)). \end{aligned}$$

The last word is in $\tilde{\Lambda}_B^2$, so it is a "good word". The second word is in $\tilde{\Lambda}_B^{21}$, the third is in $\tilde{\Lambda}_B^{12}$ and the first one is in $\tilde{\Lambda}_B^1$ but of length $2n-3$. Since ϕ is invariant with respect to conjugation by u_2 we see that $0 = \phi(\gamma_2\gamma_3 \cdots \gamma_{2n-1}\gamma_{2n}) = \phi(u_2\gamma_2\gamma_3 \cdots \gamma_{2n-1}\gamma_{2n}u_2^*) = \phi(E(u_2\gamma_2)\gamma_3 \cdots \gamma_{2n-1}E(\gamma_{2n}u_2^*))$.

Similarly if we have a word $\gamma_1 \cdots \gamma_{2n-1} \in \tilde{\Lambda}_B^1$ ($\gamma_i \in A^\circ_{1+(i-1 \mod 2)} \cap \tilde{A}_{1+(i-1 \mod 2)}$) and if the "good words" are in $\tilde{\Lambda}_B^2$ and if we conjugate with the unitary $u_1 \in W_1$ we will get

$$u_1\gamma_1\gamma_2\cdots\gamma_{2n-2}\gamma_{2n-1}u_1^* = E(u_1\gamma_1)\gamma_2\cdots\gamma_{2n-2}E(\gamma_{2n-1}u_1^*)+$$

$$+(u_1\gamma_1-E(u_1\gamma_1))\gamma_2\cdots\gamma_{2n-2}E(\gamma_{2n-1}u_1^*)+E(u_1\gamma_1)\gamma_2\cdots\gamma_{2n-2}(\gamma_{2n-1}u_1^*-E(\gamma_{2n-1}u_1^*))+$$

$$+(u_1\gamma_1 - E(u_1\gamma_1))\gamma_2\cdots\gamma_{2n-2}(\gamma_{2n-1}u_1^* - E(\gamma_{2n-1}u_1^*)).$$

Notice that the last word is from $\tilde{\Lambda}_B^1$, so it is a "good word". The second word is from $\tilde{\Lambda}_B^{12}$ and the third one is from $\tilde{\Lambda}_B^{21}$. The first word is from $\tilde{\Lambda}_B^2$ but with length $2n-3$. In this case we also can conclude that $0 = \phi(\gamma_1\gamma_2\cdots\gamma_{2n-2}\gamma_{2n-1}) = \phi(u_1\gamma_1\gamma_2\cdots\gamma_{2n-2}\gamma_{2n-1}u_1^*) = \phi(E(u_1\gamma_1)\gamma_2\cdots\gamma_{2n-2}E(\gamma_{2n-1}u_1^*))$.

From this we can conclude that if we take the word $\gamma_2\cdots\gamma_{2n} \in \tilde{\Lambda}_B^2$ and if the "good words" are in $\tilde{\Lambda}_B^1$ then $(u_1u_2)\gamma_2\cdots\gamma_{2n}(u_2^*u_1^*)$ will be the span of some "good words", i.e. belonging to $\tilde{\Lambda}_B^1$, some words from $\tilde{\Lambda}_B^{21}$, some words from $\tilde{\Lambda}_B^{12}$, and the word from $\tilde{\Lambda}_B^2$ with length $2n-5$

$$E(u_1E(u_2\gamma_2)\gamma_3)\gamma_4\cdots\gamma_{2n-2}E(\gamma_{2n-1}E(\gamma_{2n}u_2^*)u_1^*) =$$

$$= E(u_1u_2\gamma_2\gamma_3)\gamma_4\cdots\gamma_{2n-2}E(\gamma_{2n-1}\gamma_{2n}u_2^*u_1^*)$$

if $n \geq 3$. Continuing in the same fashion we see that if $l \geq n/2$, $(u_1u_2)^l\gamma_2\cdots\gamma_{2n}(u_2^*u_1^*)$ will be the span of some "good words", i.e. belonging to $\tilde{\Lambda}_B^1$, some words from $\tilde{\Lambda}_B^{21}$, some words from $\tilde{\Lambda}_B^{12}$, and a word $b \in \tilde{B}$. Actually it is easy to see that $b = E((u_1u_2)^l\gamma_2\cdots\gamma_{2n}(u_2^*u_1^*)^l) \in \tilde{B}$ since this is the element which projects onto B under the conditional expectation. Notice that since ϕ is E-invariant and also invariant with respect to conjugation by u_1 and u_2 then $\phi(E((u_1u_2)^l\gamma_2\cdots\gamma_{2n}(u_2^*u_1^*)^l)) = 0$.

We can now return to the element $x = x_B + x_1 + x_2 + x_{12} + x_{21}$. Set the words from $\tilde{\Lambda}_B^1$ to be "good words". From the observation above we see that if l is greater that the length of the longest word appearing in x_2, then $(u_1u_2)^l x_2 (u_2^*u_1^*)^l$ is the span of some "good words" from $\tilde{\Lambda}_B^1$, some words from $\tilde{\Lambda}_B^{21}$, some words from $\tilde{\Lambda}_B^{12}$, and some words from $\tilde{B}$, each one of them when evaluated on ϕ gives 0. But considering cases (i), (ii) and (iii) we can easily conclude that $x' \overset{def}{=} (u_1u_2)^l x (u_2^*u_1^*)^l$ can be written as $x' = x'_B + x'_1 + x'_{12} + x'_{21}$ with x'_B being a span of words from $\tilde{B}$ and satisfying $\phi(x'_B) = 0$, x'_1 being a span of "good words" from $\tilde{\Lambda}_B^1$, x'_{12} being a span of words from $\tilde{\Lambda}_B^{12}$ and x'_{21} being a span of words from $\tilde{\Lambda}_B^{21}$.

Let $x'_B = \sum_{i=1}^{n}\alpha_i b_i$, where $b_i \in \tilde{B}$ and $\alpha_i \in \mathbb{C}$. $0 = \phi(x'_B) = \phi(\sum_{i=1}^{n}\alpha_i b_i) = \sum_{i=1}^{n}\alpha_i\phi(b_i)$. Thus $x'_B = \sum_{i=1}^{n}\alpha_i(b_i - \phi(b_i))$ if we set $b'_i = b_i - \phi(b_i)$ for $i = 1, \ldots, n$, then $b'_i \in \tilde{B}$ with $\phi(b'_i) = 0 = \phi(u_2 b_i u_2^*)$. So we can apply condition (13) to the set of elements

$\{b_1',\dots,b_n',E(u_2b_1'u_2^*),\dots,E(u_2b_n'u_2^*)\}\subset\tilde{B}$. We obtain unitaries $\nu_1,\dots,\nu_m$. Set

$$u=\begin{cases}\nu_1\cdots\nu_m, & \text{if } \nu_1\in W_2,\nu_m\in W_2\\ u_2\nu_1\cdots\nu_m, & \text{if } \nu_1\in W_1,\nu_m\in W_2,\\ u_2\nu_1\cdots\nu_m u_2, & \text{if } \nu_1\in W_1,\nu_m\in W_1,\\ \nu_1\cdots\nu_m u_2, & \text{if } \nu_1\in W_2,\nu_m\in W_1.\end{cases}$$

Then it is clear that $u^*x_B'u\in\tilde{\Lambda}_B^2$ and the "good words" are in $\tilde{\Lambda}_B^2$. Then from cases (ii) and (iii) also follows that $x''\stackrel{def}{=}u^*x'u$ can be represented as $x''=x_2''+x_{12}''+x_{21}''$, where $x_2''\in\tilde{\Lambda}_B^2$ is a span of "good words" and $x_{12}''\in\tilde{\Lambda}_B^{12}$, $x_{21}''\in\tilde{\Lambda}_B^{21}$.
Let n be the number of words from $\tilde{\Lambda}_B^{21}$ and from $\tilde{\Lambda}_B^{12}$ that appear in the span of $x_{12}''+x_{21}''$. We will argue by induction on n to conclude the proof of the lemma.
Let $\gamma_1\cdots\gamma_{2l}\in\tilde{\Lambda}_B^{12}$ ($\gamma_i\in A^\circ_{1+(i-1 \mod 2)}\cap\tilde{A}_{1+(i-1 \mod 2)}$) is a word from the span of x_{12}''. (The case x_{21}'' is completely analoguous.) Set

$$\tilde{u}\stackrel{def}{=}\begin{cases}\omega_1\omega_2(u_1u_2)^{l-1}, & \text{if } \omega_2\in W_2,\\ \omega_1(u_2u_1)^{l-1}u_2, & \text{if } \omega_2=1_A.\end{cases}$$

Let's observe first that if $\alpha_1\cdots\alpha_{2l},\beta_1\cdots\beta_{2l}\in\tilde{\Lambda}_B^{12}$, then we can write

$$E(\beta_{2l}^*\cdots\beta_2^*\beta_1^*\alpha_1\alpha_2\cdots\alpha_{2l})=E(\beta_{2l}^*\cdots\beta_2^*E(\beta_1^*\alpha_1)\alpha_2\cdots\alpha_{2l})+$$
$$+E(\beta_{2l}^*\cdots\beta_2^*(\beta_1^*\alpha_1-E(\beta_1^*\alpha_1))\alpha_2\cdots\alpha_{2l})=E(\beta_{2l}^*\cdots\beta_2^*E(\beta_1^*\alpha_1)\alpha_2\cdots\alpha_{2l}).$$

It follows by induction that $E(\beta_{2l}^*\cdots\beta_2^*\beta_1^*\alpha_1\alpha_2\cdots\alpha_{2l})\in\tilde{B}$. Also from

$$\beta_{2l}^*\cdots\beta_2^*\beta_1^*\alpha_1\alpha_2\cdots\alpha_{2l}=\beta_{2l}^*\cdots\beta_2^*E(\beta_1^*\alpha_1)\alpha_2\cdots\alpha_{2l}+$$
$$+\beta_{2l}^*\cdots\beta_2^*(\beta_1^*\alpha_1-E(\beta_1^*\alpha_1))\alpha_2\cdots\alpha_{2l}=\beta_{2l}^*\cdots\beta_2^*E(\beta_1^*\alpha_1)\alpha_2\cdots\alpha_{2l}$$

again by induction follows that $\beta_{2l}^*\cdots\beta_2^*\beta_1^*\alpha_1\alpha_2\cdots\alpha_{2l}$ is the span of words from $\tilde{\Lambda}_B^2$ plus the word $E(\beta_{2l}^*\cdots\beta_2^*\beta_1^*\alpha_1\alpha_2\cdots\alpha_{2l})\in\tilde{B}$.

All this implies that $\tilde{u}^*\gamma_1\cdots\gamma_{2l}\tilde{u}$ is a span of "good words" from $\tilde{\Lambda}_B^2$ and the word $E(\tilde{u}^*\gamma_1\cdots\gamma_{2l})\tilde{u}\in\tilde{\Lambda}_B^{12}$. Set $\tilde{b}\stackrel{def}{=}E(\tilde{u}^*\gamma_1\cdots\gamma_{2l})\in\tilde{B}$ (see the observation above). Now we choose unitaries $\omega_1^{\tilde{b}},\omega_2^{\tilde{b}}$ as in the statement of the lemma. We have

$$(\omega_2^{\tilde{b}})^*(\omega_1^{\tilde{b}})^*E(\tilde{u}^*\gamma_1\cdots\gamma_{2l})\tilde{u}\omega_1^{\tilde{b}}\omega_2^{\tilde{b}}=$$
$$=\begin{cases}(\omega_2^{\tilde{b}})^*(\omega_1^{\tilde{b}})^*E(\tilde{u}^*\gamma_1\cdots\gamma_{2l})\omega_1\omega_2(u_1u_2)^{l-1}\omega_1^{\tilde{b}}\omega_2^{\tilde{b}}, & \text{if } \omega_2\in W_2,\\ (\omega_1^{\tilde{b}})^*E(\tilde{u}^*\gamma_1\cdots\gamma_{2l})\omega_1(u_2u_1)^{l-1}u_2\omega_1^{\tilde{b}}, & \text{if } \omega_2=1_A.\end{cases}$$

From this and from the choice of $\omega_1^{\tilde{b}},\omega_2^{\tilde{b}}$ (and from case (i)) it is clear that

$$(\omega_2^{\tilde{b}})^*(\omega_1^{\tilde{b}})^*E(\tilde{u}^*\gamma_1\cdots\gamma_{2l})\tilde{u}\omega_1^{\tilde{b}}\omega_2^{\tilde{b}}$$

is a span of "good words".

Since by cases (ii) and (iii) follows that when we alternatively conjugate words from $\tilde{\Lambda}_B^{12}$ and from $\tilde{\Lambda}_B^{21}$ by unitaries from W_1 and W_2 the number of such words doesn't increase, and since we managed to conjugate the word $\gamma_1\cdots\gamma_{2l}$ to a span of "good words", the induction on n is concluded.

This proves the lemma. $\square$

Combining Proposition 4.3 and Lemma 4.4 we obtain the following

Theorem 4.5. *Assume that we have unital C^*-algebras A_1, A_2 that contain the unital C^*-algebra B as a unital C^*-subalgebra. Also assume that there are condidional expectations $E_i : A_i \to B$ for $i = 1, 2$ that have faithful KSGNS-rapresentations (i.e. satisfy condition (1)) and form the reduced amalgamated free product $(A, E) \stackrel{def}{=} (A_1, E_1) * (A_2, E_2)$.*

Suppose that there are states ϕ_i on A_i for $i = 1, 2$ which are invariant with respect to E_i, $i = 1, 2$ and satisfy $\phi_1|_B = \phi_2|_B (\stackrel{def}{=} \phi_B)$. Construct $\phi \stackrel{def}{=} \phi_B \circ E$. Assume that there are unitaries $u_1 \in A_1$, $u_2, u_2' \in A_2$ with $E_1(u_1) = 0 = E_2(u_2) = E_2(u_2') = E(u_2 u_2'^*)$. (Or assume that there are unitaries $u_1, u_1' \in A_1^\circ$, $u_2 \in A_2^\circ$ with $E(u_1 u_1'^*) = 0$.)

Suppose that there are two multiplicative sets $1_A \in \tilde{A}_i \subset A_i$ such that $\mathrm{Span}(\tilde{A}_i)$ is dense in A_i, suppose from $a_i \in \tilde{A}_i$ follows $E_i(a_i)$, $a_i - E_i(a_i)$, $a_i - \phi_i(a_i) \in \tilde{A}_i$, for $i = 1, 2$, and $B \cap \tilde{A}_1 = B \cap \tilde{A}_2 \stackrel{def}{=} \tilde{B}$.

Suppose also that there are two sets of unitaries $\emptyset \neq W_i \subset \tilde{A}_i \cap A_i^\circ$ such that $(W_i)^* \subset \tilde{A}_i$ for $i = 1, 2$. Let $v_i \in W_i$, $i = 1, 2$ and suppose that ϕ is invariant with respect to conjugation by v_1 and v_2.

Suppose that condition (13) holds, namely:

$$\begin{cases} \forall b_1, \dots, b_l \in \tilde{B},\ \text{with } \phi(b_1) = \cdots = \phi(b_l) = 0,\ \exists m \in \mathbb{N} \text{ and unitaries} \\ \nu_{11}, \dots, \nu_{1m}, \nu_{21}, \dots, \nu_{2m} \text{ such that } \nu_{12}, \dots, \nu_{1m} \in W_1,\ \nu_{21}, \dots, \nu_{2(m-1)} \in W_2,\ \text{and:} \\ \text{either } \nu_{11} \in W_1,\ \nu_{2m} \in W_2 \text{ or} \\ \nu_{11} = 1_{A_1},\ \nu_{2m} \in W_2,\ \text{or} \\ \nu_{11} \in W_1,\ \nu_{2m} = 1_{A_2}, \\ \nu_{11} = 1_{A_1},\ \nu_{2m} = 1_{A_2},\ k \geq 2 \\ \text{with } E((\nu_{11}\nu_{21}\nu_{12} \cdots \nu_{1m}\nu_{2m}) b_k (\nu_{11}\nu_{21}\nu_{12} \cdots \nu_{1m}\nu_{2m})^*) = 0 \text{ for } k = 1, \dots, l, \\ (\text{i.e. there are unitaries that conjugate } \tilde{B} \ominus \mathbb{C}1_B \text{ out of } B) \end{cases}$$

Suppose finally that there are unitaries $\omega_1 \in W_1$ and ω_2 with $\omega_2 = 1_A$ or $\omega_2 \in W_2$, such that $\forall b \in \tilde{B}$, $\exists \omega_1^b \in W_1$, and $\omega_2^b \in W_2$ if $\omega_2 \in W_2$ or $\omega_2^b = 1$ if $\omega_2 = 1$ with $E((\omega_2^b)^* (\omega_1^b)^* b \omega_1 \omega_2) = 0$.

Then:

(1) If ϕ_B has a faithful GNS-representation then A is simple.

(2) If ϕ is invariant with respect to conjugation by u_1, u_2, u_2' (or by u_1, u_1', u_2) and all the unitaries from W_1 and W_2, then ϕ is the only tracial state on A, invariant with respect to conjugation by all those unitaries.

Proof. (1) Suppose $I \neq 0$ is an ideal of A. Notice that $\mathrm{Alg}(\tilde{A}_1 \cup \tilde{A}_2)$ is dense in A. Take a nonzero element $x \in I$. Because E has a faithful KSGNS-representation it satisfies condition (1), i.e. $\exists y \in A$ such that $b \stackrel{def}{=} E(y^* x^* x y) \neq 0$. Notice that $b^* = b$. Since ϕ_B has a faithful GNS-representation we can find $b' \in B$ such that

$\phi_B((b')^*bb') \neq 0$. Then

$$\phi((b')^*y^*x^*xyb') = \phi(E((b')^*y^*x^*xyb')) = \phi((b')^*E(y^*x^*xy)b') = \phi((b')^*bb') \neq 0.$$

Then $c \overset{def}{=} \phi((b')^*bb')^{-1}(b')^*y^*x^*xyb'$ $(\in I)$ is self-adjoined and satisfies $\phi(c) = 1$. Find $a \in \mathrm{Alg}(\tilde{A}_1 \cup \tilde{A}_2)$ such that $\|a - c\| < \epsilon$. From Lemma 4.4 it follows that we can find unitaries $\alpha_1, \ldots, \alpha_m \in W_1 \cup W_2$ such that $(\alpha_1 \cdots \alpha_m)^*(a - \phi(a)1_A)(\alpha_1 \cdots \alpha_m) \in \Lambda_B^1$. Then it follows from Proposition 4.3 that we can find unitaries $U_1, \ldots U_N \in A$ that are constructed from u_1, u_2, u_2' and the unitaries from $W_1 \cup W_2$ and are such that

$$\|\sum_{i=1}^{N} \frac{1}{N} U_i^*(\alpha_1 \cdots \alpha_m)^*(a - \phi(a)1_A)(\alpha_1 \cdots \alpha_m)U_i\| < \epsilon.$$

Then

$$\|\sum_{i=1}^{N} \frac{1}{N} U_i^*(\alpha_1 \cdots \alpha_m)^*(a - \phi(a)1_A - c + 1_A)(\alpha_1 \cdots \alpha_m)U_i\| \leq$$
$$\sum_{i=1}^{N} \frac{1}{N} \|U_i^*(\alpha_1 \cdots \alpha_m)^*(a - \phi(a)1_A - c + 1_A)(\alpha_1 \cdots \alpha_m)U_i\| =$$
$$= \sum_{i=1}^{N} \frac{1}{N} \|a - \phi(a)1_A - c + 1_A\| = \|a - \phi(a)1_A - c + 1_A\| =$$
$$= \|(a - c) - \phi(a - c)\| \leq \|a - c\| + \|a - c\| < 2\epsilon.$$

Therefore $\|\sum_{i=1}^{N} \frac{1}{N} U_i^*(\alpha_1 \cdots \alpha_m)^*(c - 1_A)(\alpha_1 \cdots \alpha_m)U_i\| < 3\epsilon$. Set $d \overset{def}{=} \sum_{i=1}^{N} \frac{1}{N} U_i^*(\alpha_1 \cdots \alpha_m)^*c(\alpha_1 \cdots \alpha_m)U_i$ $(\in I)$.

Thus $\|d - 1_A\| < 3\epsilon$. Then if we take $\epsilon < \frac{1}{3}$ it would follow that d is invertible, and therefore $I = A$.

(2) Take $0 \neq x \in A$. Then if we argue as in case (1) we can find unitaries $U_1, \ldots, U_N \in \overline{conv}\{u | u$ is a product of unitaries from $W_1 \cup W_2 \cup \{u_1, u_2, u_2'\}\}$ with

$$\|\sum_{i=1}^{N} \frac{1}{N} U_i^*(x - \phi(x)1_A)U_i\| < 3\epsilon.$$

If we take a state ϕ' such that ϕ and ϕ' are invariant with respect to conjugation by u_1, u_2, u_2' and by all unitaries from $W_1 \cup W_2$ then we will have

$$3\epsilon > |\phi'(\sum_{i=1}^{N} \frac{1}{N} U_i^*(x - \phi(x)1_A)U_i)| = |\sum_{i=1}^{N} \frac{1}{N} \phi'(U_i^*xU_i) - \phi(x)| = |\sum_{i=1}^{N} \frac{1}{N} \phi'(x) - \phi(x)| =$$
$$|\phi'(x) - \phi(x)|.$$

Since this is true for any $\epsilon > 0$ it follows that $\phi' \equiv \phi$. □

Although the statement of Theorem 4.5 looks complicated some applications can be given. The next proposition is a slight generalization of the de la Harpe's result from [9].

Corollary 4.6. *Suppose that $G_1 \supsetneq H \subsetneq G_2$ are discrete groups and suppose that $H \neq \{1\}$. Denote $G \stackrel{def}{=} G_1 \underset{H}{*} G_2$. Suppose that for any finitely many $h_1, \ldots, h_m \in H \backslash \{1\}$ there is $g \in G$ with $g^{-1} h_i g \notin H$ for all $i = 1, \ldots, m$. Then $C_r^*(G)$ is simple with a unique trace.*

Proof. Set $A_i = C_r^*(G_i)$, $i = 1, 2$, $B = C_r^*(H)$ and $A = C_r^*(G)$. Clearly H is not normal in at least one of the groups G_1 or G_2. Without loss of generality suppose that H is not normal in G_1. Then there are $g_1, g_1' \in G_1 \backslash H$ and $g_2 \in G_2 \backslash H$ with $g_1 (g_1')^{-1} \in G_1 \backslash H$. Then set $u_1 = \lambda(g_1)$, $u_1' = \lambda(g_1')$, $u_2 = \lambda(g_2)$. We take $\tilde{A}_i = \{\lambda(c_i) | c_i \in G_i\}$, $i = 1, 2$, $\tilde{B} = \{\lambda(h) | h \in H\}$. Also $W_i = \tilde{A}_i \backslash \tilde{B}$ for $i = 1, 2$. Condition (13) is satisfied since for finitely many elements from $H \backslash \{1\}$ we can find an element from G that conjugates them away from H. Finally for the last condition of Theorem 4.5 we can set $\omega_1 = u_1$, $\omega_2 = 1$ and for $\lambda(h) \in \tilde{B}$ we set $\omega_1^{\lambda(h)} = h u_1'$. Thus all requirements of Theorem 4.5 are met and this finishes the proof. □

We give also an application to HNN extensions of discrete groups. We will use the notion of reduced HNN extensions for C^*-algebras introduced by Ueda in [16]. We will use the following settings:

Let $\{1\} \subsetneq H \subset G$ be countable discrete groups and let $\tilde{\theta} : H \to G$ be an injective group homomorphism. Thus we have that $C_r^*(H) \subset C_r^*(G)$ and that we have a well defined injective $*$-homomorphism $\theta : C_r^*(H) \to C_r^*(G)$. By $E_H^G : C_r^*(G) \to C_r^*(H)$ and $E_{\tilde{\theta}(H)}^G : C_r^*(G) \to C_r^*(\theta(C_r^*(H))$ we will denote the canonical conditional expectations. By τ_G we will denote the canonical trace on $C_r^*(G)$. Let $A_1 = C_r^*(G) \otimes M_2(\mathbb{C})$, $A_2 = C_r^*(H) \otimes M_2(\mathbb{C})$ and $B = C_r^*(H) \oplus C_r^*(H)$. Define the inclusion maps $i_1 : B \to A_1$ and $i_2 : B \to A_2$ as

$$i_1(b_1 \oplus b_2) = \begin{bmatrix} b_1 & 0 \\ 0 & \theta(b_2) \end{bmatrix}, \qquad i_2(b_1 \oplus b_2) = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$$

and define the conditional expectations $E_1 : A_1 \to B$ and $E_2 : A_2 \to B$ as

$$E_1 = \begin{bmatrix} E_H^G & 0 \\ 0 & E_{\tilde{\theta}(H)}^G \end{bmatrix}, \qquad E_2 = \begin{bmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{bmatrix}.$$

Then let

$$(A, E) = (A_1, E_1) * (A_2, E_2)$$

be the reduced amalgamated free product of (A_1, E_1) and (A_2, E_2) and let

$$(\mathcal{A}, E_{C_r^*(G)}^{\mathcal{A}}, u(\theta)) = (C_r^*(G), E_H^G) \bigstar_{C_r^*(H)} (\theta, E_{\tilde{\theta}(H)}^G)$$

be the reduced HNN extension of $C_r^*(G)$ by θ as in [16]. Also let $i_B : B \to A$ be the canonical inclusion.

From [16, Proposition 2.2] follows that A is isomorphic to $\mathcal{A} \otimes M_2(\mathbb{C})$. Therefore the questions of simplicity and uniqueness of trace for A and for $\mathcal{A}$ are equivalent. The following corollary of Theorem 4.5 is true:

Corollary 4.7. *In the above settings suppose that $H \subsetneq G$ and $\tilde{\theta}(H) \subsetneq G$. Suppose also that $\forall h \in H \backslash \{1\}$, $\exists n_h \in \mathbb{N}$, such that $\tilde{\theta}^{n_h - 1}(h) \in H$ and $\tilde{\theta}^{n_h}(h) \notin H$. Then A (and therefore $\mathcal{A}$ also) is simple with a unique trace.*

Proof. We will show that all the conditions of Theorem 4.5 are met.

First the canonical traces τ_i on A_i, $i = 1,2$ satisfy $\tau_i \circ E_i = \tau_i$ for $i = 1,2$ and $\tau_1|_B = \tau_2|_B$ ($\stackrel{def}{=} \tau_B$). We have $\tau = \tau_B \circ E$.

Define

$$\tilde{A}_1 = \mathrm{Span}(\{\lambda(g) \otimes e_{ij} | g \in G,\ 1 \le i,j \le 2\})$$

and

$$\tilde{A}_2 = \mathrm{Span}(\{\lambda(h) \otimes e_{ij} | h \in H,\ 1 \le i,j \le 2\}),$$

where e_{ij} for $1 \le i,j \le 2$ are the matrix units for $M_2(\mathbb{C})$. Then we have $\tilde{A}_1 \cap B = \tilde{A}_2 \cap B (\stackrel{def}{=} \tilde{B})$. It is also clear that $a_i \in \tilde{A}_i$ implies $E(a_i),\ a_i - E(a_i), a_i - \tau_i(a_i) \in \tilde{A}_i$ for $i = 1,2$.

Choose $\bar{g}_1 \in G \backslash H,\ \bar{g}_2 \in G \backslash \tilde{\theta}(H)$.

Define the following unitaries from $A_1 \cap \tilde{A}_1$:

$$u_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad u_1' = \begin{bmatrix} \lambda(\bar{g}_1) & 0 \\ 0 & \lambda(\bar{g}_2) \end{bmatrix}, \qquad u_1'' = \frac{1}{\sqrt{2}} \begin{bmatrix} -\lambda(\bar{g}_1) & \lambda(\bar{g}_1) \\ \lambda(\bar{g}_2) & \lambda(\bar{g}_2) \end{bmatrix},$$

and the following unitary from $A_2 \cap \tilde{A}_2$:

$$u_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Set $W_1 = \{u_1, u_1', u_1''\}$, $W_2 = \{u_2\}$.

Set $\omega_1 = u_1$, $\omega_2 = 1_{A_2}$ and for every $b = b_1 \oplus b_2 \in \tilde{B}$ set $\omega_1^b = u_1'$. Then

$$E((u_1')^*(b_1 \oplus b_2)u_1) = E(\begin{bmatrix} \lambda(\bar{g}_1^{-1}) & 0 \\ 0 & \lambda(\bar{g}_2^{-1}) \end{bmatrix} \begin{bmatrix} b_1 & 0 \\ 0 & \theta(b_2) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) =$$

$$= E(\begin{bmatrix} 0 & \lambda(\bar{g}_1^{-1})b_1 \\ \lambda(\bar{g}_1^{-1})\theta(b_2) & 0 \end{bmatrix}) = 0.$$

It remains to check that condition (13) holds.

For an element $b = b_1 \oplus b_2 \in B$ it is easy to see that

$$u_2^* u_1^* b u_1 u_2 = E(u_2^* E(u_1^* b u_1) u_2) + u_2^*(u_1^* b u_1 - E(u_1^* b u_1))u_2$$

and that

$$i_B^{-1} \circ E(u_2^* u_1^* b u_1 u_2) = \begin{cases} \theta^{-1}(b_1) \oplus \theta(b_2), & \text{if } b_1 \in \theta(C_r^*(H)),\ b_2 \in C_r^*(H), \\ \theta^{-1}(b_1) \oplus 0, & \text{if } b_1 \in \theta(C_r^*(H)),\ b_2 \notin C_r^*(H), \\ 0 \oplus \theta(b_2), & \text{if } b_1 \notin \theta(C_r^*(H)),\ b_2 \in C_r^*(H), \\ 0 \oplus 0, & \text{if } b_1 \notin \theta(C_r^*(H)),\ b_2 \notin C_r^*(H). \end{cases}$$

Using induction one can show that for any $n \in \mathbb{N}$ we have

$$(u_2^* u_1^*)^n b (u_1 u_2)^{-n} - E((u_2^* u_1^*)^{-n} b (u_1 u_2)^n) \in \Lambda_B^2.$$

Let $\hat{\theta}$ be the linear map which extends θ to $C_r^*(G)$ by $\hat{\theta}(\lambda(g)) = 0$ for $g \in G \backslash H$. Also let $\hat{\theta^{-1}}$ be the linear map which extends θ^{-1} to $C_r^*(G)$ by $\hat{\theta^{-1}}(\lambda(g)) = 0$ for $g \in G \backslash \tilde{\theta}(H)$. Then:

$$i_B^{-1} \circ E((u_2^* u_1^*)^{-n} b (u_1 u_2)^n) =$$

$$= \begin{cases} \theta^{-n}(b_1) \oplus \theta^n(b_2), & \text{if } b_1 \in (\hat{\theta})^n(C_r^*(H)),\ b_2 \in (\hat{\theta^{-1}})^{n-1}(C_r^*(H)), \\ \theta^{-n}(b_1) \oplus 0, & \text{if } b_1 \in (\hat{\theta})^n(C_r^*(H)),\ b_2 \notin (\hat{\theta^{-1}})^{n-1}(C_r^*(H)), \\ 0 \oplus \theta^n(b_2), & \text{if } b_1 \notin (\hat{\theta})^n(C_r^*(H)),\ b_2 \in (\hat{\theta^{-1}})^{n-1}(C_r^*(H)), \\ 0 \oplus 0, & \text{if } b_1 \notin (\hat{\theta})^n(C_r^*(H)),\ b_2 \notin (\hat{\theta^{-1}})^{n-1}(C_r^*(H)). \end{cases}$$

If we set $c_1 = \lambda(\bar{g}_1^{-1})(\hat{\theta^{-1}})^n(b_1)\lambda(\bar{g}_1)$ and $c_2 = \lambda(\bar{g}_2^{-1})(\hat{\theta})^{n+1}(b_2)\lambda(\bar{g}_2)$ the we will have

$$i_B^{-1} \circ E(u_2^*(u_1')^*(u_2^*u_1^*)^n b(u_1u_2)^n u_1'u_2) =$$

$$= \begin{cases} \theta^{-1}(c_2) \oplus c_1, & \text{if } c_2 \in \theta(C_r^*(H)),\ c_1 \in C_r^*(H), \\ \theta^{-1}(c_2) \oplus 0, & \text{if } c_2 \in \theta(C_r^*(H)),\ c_1 \notin C_r^*(H), \\ 0 \oplus c_1, & \text{if } c_2 \notin \theta(C_r^*(H)),\ c_1 \in C_r^*(H), \\ 0 \oplus 0, & \text{if } c_2 \notin \theta(C_r^*(H)),\ c_1 \notin C_r^*(H). \end{cases}$$

Now take elements $\tilde{b}_1, \ldots, \tilde{b}_l \in \tilde{B}$ with $\tau_B(\tilde{b}_1) = \cdots = \tau_B(\tilde{b}_l) = 0$. We can write $\tilde{b}_k = \alpha_k + b_{k1} \oplus -\alpha_k + b_{k2}$ for each $k = 1, \ldots, l$ with $b_{kj} \in \mathrm{Span}(\{\lambda(h) | h \in H\backslash\{1\}\})$. Clearly from the statement of the corollary follows that there exists an $N \in \mathbb{N}$ with $E_H^G(\hat{\theta}^N(b_{k2})) = 0$ for each $k = 1, \ldots, l$. Therefore for each $k = 1, \ldots, l$ we have

$$i_B^{-1} \circ E(u_2^*(u_1')^*(u_2^*u_1^*)^{-N}\tilde{b}_k(u_1u_2)^N u_1'u_2) =$$

$$= \begin{cases} \alpha_k \oplus -\alpha_k + c_k, & \text{if } c_k \in C_r^*(H), \\ \alpha_k \oplus -\alpha_k, & \text{if } c_k \notin C_r^*(H), \end{cases}$$

where $c_k = \lambda(\bar{g}_1^{-1})(\hat{\theta^{-1}})^N(b_{k1})\lambda(\bar{g}_1)$, $k = 1, \ldots l$. Now we can find an $M \in \mathbb{N}$ such that $(\hat{\theta})^M(c_k) = 0$ for all $k = 1, \cdots, l$. Then for all $k = 1, \ldots, l$ we have

$$i_B^{-1} \circ E((u_2^*u_1^*)^{-M} u_2^*(u_1')^*(u_2^*u_1^*)^{-N}\tilde{b}_k(u_1u_2)^N u_1'u_2(u_1u_2)^M) = \alpha_k \oplus -\alpha_k.$$

Finally for all $k = 1, \ldots, l$

$$i_B^{-1} \circ E((u_1'')^*(u_2^*u_1^*)^{-M} u_2^*(u_1')^*(u_2^*u_1^*)^{-N}\tilde{b}_k(u_1u_2)^N u_1'u_2(u_1u_2)^M u_1'') = 0.$$

This proves that condition (13) holds and thus we can apply Theorem 4.5.

This proves the Corollary. □

Remark 4.8. *By symmetry it is clear that in the corollary the assumption*
" $\forall h \in H\backslash\{1\}$, $\exists n_h \in \mathbb{N}$, such that $\tilde{\theta}^{n_h-1}(h) \in H$ and $\tilde{\theta}^{n_h}(h) \notin H$ "
can be replaced by the assumption
" $\forall h \in H\backslash\{1\}$, $\exists n_h \in \mathbb{N}$, such that $\tilde{\theta}^{-n_h+1}(h) \in \tilde{\theta}(H)$ and $\tilde{\theta}^{-n_h}(h) \notin \tilde{\theta}(H)$ ".

Examples of HNN extensions of discrete groups which satisfy the assumption of this corollary (and whose reduced C^*-algebras are simple with a unique trace) are the Baumslag-Solitar groups $BS(n,m) \stackrel{def}{=} \langle a, b \mid b^{-1}a^m b = a^n \rangle$ for $|n| \neq |m|$ and $|n|, |m| \geq 2$.

Somewhat related result is the ICC property. It was proved by Stalder in [15] that $BS(m,n)$ is an ICC group if and only if $|n| \neq |m|$.

Prof. Ueda pointed out to me that our result on the C^*-simplicity of $BS(m,n)$ is sharp:

In the case $m = 1$ (or $n = 1$) it is known that those groups are solvable. $B(1,1)$ is abelian. For $|n| > 1$ $BS(1,n) = \langle a, b \mid b^{-1}ab = a^n \rangle$. It is not difficult to see that all the elements of $BS(1,n)$ can be written in the form $b^i a^k b^{-j}$, where $i, j \geq 0$ and if $ij > 0$ then $n \nmid k$. Then $b^i a^k b^{-j} \mapsto i - j$ is a well defined group homomorphism $h : BS(1,n) \to \mathbb{Z}$. Then one can check that $\ker(h) = \langle b^i a^k b^{-i},\ i \geq 0,\ k \in \mathbb{Z} \mid b^{i+1} a^{nk} b^{-i-1} = b^i a^k b^{-i} \rangle$ and it is isomorphic to the additive group of the n-adic numbers. This shows that $BS(1,n)$ is meta-abelian (extension of an abelian by an abelian) group and trerefore solvable. It is also know that extension of an amenable group by an amenable group is amenable group and therefore the solvable groups are amenable.

If G is a locally compact discrete group then we have (by definition) the one-dimensional representation of the full C^*-algebra of G $\pi : C^*(G) \to \mathbb{C}$ given on the generators of G by $\pi(f_g) = 1$ for all $g \in G$ ($f_g : G \to \mathbb{C}$ is given by $f_g(h) = \delta_{gh},\ h \in G$). If $|G| > 1$ $\ker(\pi)$ is a nontrivial ideal in $C^*(G)$. Obviously π is a tracial state. If $|G| > 1$ then $1 = \pi(f_g) \neq \tau_G(f_g) = 0$ for $\forall 1 \neq g \in G$, where τ_G is the canonical trace on $C^*(G)$. Therefore if $|G| > 1$ then $C^*(G)$ has more that one trace. All this shows that if G is an amenable locally compact discrete group and if $|G| > 1$ then $C^*_r(G)$ $(= C^*(G))$ is not simple and has more than one trace. Therefore $C^*_r(BS(1,n))$ are not simple and each one has more than one trace for each $n \in \mathbb{Z}$.

Finally if $m = n$ and $|n|, |m| \geq 2$ then $BS(n,n)$ has a nontrivial center (a^n is in the center of $BS(n,n)$). If $m = -n$ then $C^*_r(BS(-n,n))$ has a nontrivial center ($\lambda(a^n) + \lambda(a^{-n})$ is in the center of $C^*_r(BS(-n,n))$). In both cases $C^*_r(BS(m,n)) \cong A \otimes C(X)$ for some C^*-algebra A and some compact Hausdorff space X ($|X| > 1$). If $x \in X$ then $I = \langle a \otimes f \mid a \in A,\ f \in C(X),\ f(x) = 0 \rangle_{A \otimes C(X)}$ is a nontrivial ideal of $A \otimes C(X)$. Also if we call τ_A the restriction of the canonical trace on $C^*_r(BS(-n,n))$ to $A \otimes 1_X$ then if $x, y \in X$ are distinct points $x \neq y$ of X then $\tau_A \otimes ev_x$ and $\tau_A \otimes ev_y$ are two distinct tracial states on $A \otimes C(X)$. Here ev_x is the functional on $C(X)$ given by $ev_x(f) \equiv f(x),\ f \in C(X)$.

We record this as the following

Theorem 4.9. *The reduced C^*-algebra $C^*_r(BS(m,n))$ of the Baumslag-Solitar group $BS(m,n)$ is simple if and only if it has a unique trace, if and only if $|n|, |m| \geq 2$ and $|n| \neq |m|$.*

For more on C^*-simplicity of various groups see [10].

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